Test exam
Solutions to exercises

Introduction to Logic
Minor Logic and Computation

Exercise 1 (10 points)**.** Argue by making use of a truth table whether the following argument is valid or not. If it is not valid specify a counter-example.

$(p \wedge q) \rightarrow r, \neg q \vee r, p / \neg q$

Solution.

	p	q	r	$p \wedge q$	$(p \wedge q) \rightarrow r$	$\neg q$	$\neg q \vee r$	$\neg q$
1.	**1**	1	1	1	**1**	0	**1**	**0**
2.	1	1	0	1	0	0	0	0
3.	1	0	1	0	1	1	1	1
4.	1	0	0	0	1	1	1	1
5.	0	1	1	0	1	0	1	0
6.	0	1	0	0	1	0	0	0
7.	0	0	1	0	1	1	1	1
8.	0	0	0	0	1	1	1	1

This argument is not valid. In order for an argument to be valid, it must be the case that in all cases where the premises are true, the conclusion is true as well.

A counter-example is given in the first row: $V_1(p) = V_1(q) = V_1(r) = 1$. In V_1 all premises are true and the conclusion is false.

Exercise 2 (10 points)**.** Prove that $\varphi \rightarrow \neg\psi$ is a contradiction iff φ and ψ are both tautologies.

Solution. We need to prove a bi-conditional statement and therefore we need to prove both directions.

"$\Rightarrow$" Assume that $\varphi \rightarrow \neg\psi$ is a contradiction. This means that for all valuations V the valuation of the formula is false. A conditional can only be false if the antecedent is true and the consequent is false. This means that for all valuations V the antecedent of the conditional is one, $V(\varphi) = 1$, and the consequent of the conditional is false, $V(\neg\psi)$. Since φ is true for all valuations it follows by definition that φ is a tautology. Since $\neg\psi$ is false for all valuations it must be the case that ψ is true for all valuations. It follows by definition that ψ is a tautology.

"$\Leftarrow$" Assume that φ and ψ are tautologies. This means that for all valuations V: $V(\varphi) = V(\psi) = 1$. From this it follows that for all valuations V $V(\neg\psi) = 0$. Since for all valuations V we have that φ is true and $\neg\psi$ is false, we may conclude by the truth-table of the implication that for all valuations V the implication $\varphi \to \neg\psi$ is false. By definition, this means that $\varphi \to \neg\psi$ is a contradiction.

Exercise 3 (15 points)**.** Translate the following sentences in the language of first-order predicate logic. Use the identity sign if necessary.

(1) All students who passed the exam are pleased with themselves.

(2) All students made at least two exams in this semester.

(3) There is one exam that only John passed.

Solution. Other translations also possible.

Translation key: $Sx := x$ is a student, $j :=$ John, P_1xy: x passes y, P_2xy: x is pleased with y, Ex: x is an exam, $e :=$ the exam, Mxy: x made y in this semester.

(4) $\forall x((Sx \wedge P_1xe) \to P_2xx)$

(5) $\forall x(Sx \to \exists y\exists z(y \neq z \wedge Ey \wedge Ez \wedge Mxy \wedge Mxz))$

(6) $\exists x(Ex \wedge \forall y((Ey \wedge P_1jy) \leftrightarrow x = y))$

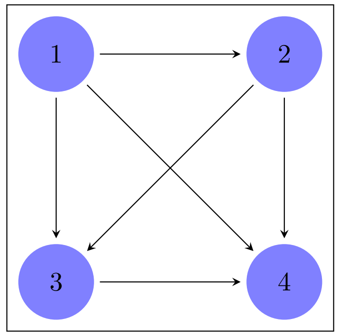
Exercise 4 (15 points)**.** Consider the model $\mathcal{M} = \langle D, I \rangle$, where

$D = \{1, 2, 3, 4\}$
$I(R) = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 3,4\rangle\}$
$I(a) = 1; I(b) = 2; I(c) = 3; I(d) = 4$

Argue whether the following sentences are true in this model or not.

1. $\forall y(\exists xRxy \leftrightarrow \exists zRyz)$
2. $\forall x\forall y\forall z((Rxy \wedge Ryz) \to Rxz)$
3. $\forall x\forall y(x = y \leftrightarrow (Rxy \leftrightarrow Ryx))$

Solution.



1. $\mathcal{M} \not\models \forall y(\exists x Rxy \leftrightarrow \exists z Ryz)$. This formula means that every point has an arrow pointing towards it iff it has a arrow pointing away from it. Object 1 in the domain is a case for which this does not hold. It has arrows leaving this point but none coming in. The statement is therefore false in this model.

2. $\mathcal{M} \models \forall x \forall y \forall z((Rxy \wedge Ryz) \rightarrow Rxz)$. This formula expresses that R is transitive: whenever there is an arrow from one point to a second and from the second to a third, there is also an arrow from the first to the third point. This formula is true in this model.

3. $\mathcal{M} \models \forall x \forall y(x = y \leftrightarrow (Rxy \leftrightarrow Ryx))$. The formula expresses that there are no arrows going back and forth between two different points. True in this model.

Exercise 5 (20 points)**.** One of the following arguments is valid, the other is invalid.

$$\forall x(Ax \vee Bx),\ \forall x(Ax \rightarrow Cx),\ \exists x \neg Cx\ /\ \exists x Bx$$
$$\forall x \exists y \exists z(x \neq y \wedge x \neq z \wedge y \neq z \wedge Rxy \wedge Rxz)\ /\ \forall x \forall y(x \neq y \rightarrow (Rxy \vee Ryx))$$

If the argument is not valid, show it by using a counter-model. If the argument is valid, give a proof.

Solution.

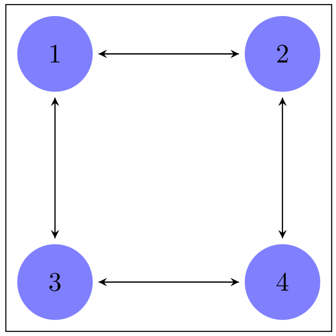
1. We prove that the first argument is valid. Assume $\mathcal{M}$ is a model in which the premises are true. This means that $\exists x \neg Cx$ is true and therefore there must be an element, let's call it o, in the domain that is not C.

   On the basis of the first premise o is A or B.

   Assume o is A. In that case on the basis of the second premise o is C. But o is not C. Contradiction. So o must be B.

   This means there is an element in the domain that is B and the conclusion is true.

2. The second argument is invalid. The premise says that for every point arrows go out to two different points. The conclusion says that between every pair of points an arrow goes between them. We define a counter-model.



Clearly, the premise is true in this model, for every point, arrows go out to two other points. But the conclusion is false. For instance there are no arrows between 1 and 4.

Exercise 6 (20 points)**.** Show by means of a natural deduction that the following assertions are correct:

1. $(p \vee (q \wedge r)) \vdash (p \vee r)$

1.	$(p \vee (q \wedge r))$	Premise
2.	p	Assumption
3.	$p \vee r$	$\mathsf{I}_\vee$, 2
4.	$p \rightarrow (p \vee r)$	$\mathsf{I}_\rightarrow$, 2, 3
5.	$q \wedge r$	Assumption
6.	r	$\mathsf{E}_\wedge$, 5
7.	$p \vee r$	$\mathsf{I}_\vee$, 6
8.	$(q \wedge r) \rightarrow (p \vee r)$	$\mathsf{I}_\rightarrow$, 5, 7
9.	$(p \vee r)$	$\mathsf{E}_\vee$, 8, 4, 1

2. $\vdash (p \vee \neg p)$

1.	$\neg(p \vee \neg p)$	Assumption
2.	p	Assumption
3.	$p \vee \neg p$	$\mathsf{I}_{\vee}$, 2
4.	$\bot$	$\mathsf{E}_{\neg}$, 3, 1
5.	$\neg p$	$\mathsf{I}_{\neg}$
6.	$p \vee \neg p$	$\mathsf{I}_{\vee}$, 5
7.	$\bot$	$\mathsf{E}_{\neg}$, 6, 1
8.	$\neg\neg(p \vee \neg p)$	$\mathsf{I}_{\neg}$
9.	$(p \vee \neg p)$	$\neg\neg$, 8

3. $\neg\exists x(Fx \wedge Gx) \vdash \forall x(Fx \rightarrow \neg Gx)$

1.	$\neg\exists x(Fx \wedge Gx)$	Premise
2.	Fa	Assumption
3.	Ga	Assumption
4.	$Fa \wedge Ga$	$\mathsf{I}_{\wedge}$, 3, 2
5.	$\exists x(Fx \wedge Gx)$	$\mathsf{I}_{\exists}$, 4
6.	$\bot$	$\mathsf{E}_{\neg}$, 5, 1
7.	$\neg Ga$	$\mathsf{I}_{\neg}$, 6
8.	$Fa \rightarrow \neg Ga$	$\mathsf{I}_{\rightarrow}$, 7, 2
9.	$\forall x(Fx \rightarrow \neg Gx)$	$\mathsf{I}_{\forall}$, 8

4. $\exists xAx \rightarrow \forall xBx, \exists x\neg Bx \vdash \neg\exists xAx$

1.	$\exists xAx \rightarrow \forall xBx$	Premise
2.	$\exists x\neg Bx$	Premise
3.	$\exists xAx$	Assumption
4.	$\forall xBx$	$\mathsf{E}_{\rightarrow}$, 3, 1
5.	$\neg Ba$	Assumption
6.	Ba	$\mathsf{E}_{\forall}$, 4
7.	$\bot$	$\mathsf{E}_{\neg}$, 6, 5
8.	$\neg Ba \rightarrow \bot$	$\mathsf{I}_{\rightarrow}$, 7, 2
9.	$\bot$	$\mathsf{E}_{\exists}$, 8, 2
10.	$\neg\exists xAx$	$\mathsf{I}_{\neg}$

5. $(\exists x Rax \wedge \forall x Rxa) \to \forall x Rax, \forall x Rxa \vdash Rab$

1.	$(\exists x Rax \wedge \forall x Rxa) \to \forall x Rax$	Premise
2.	$\forall x Rxa$	Premise
3.	Raa	$\mathsf{E}_\forall$, 2
4.	$\exists x Rax$	$\mathsf{I}_\exists$, 3
5.	$\exists x Rax \wedge \forall x Rxa$	$\mathsf{I}_\wedge$, 4, 2
6.	$\forall x Rax$	$\mathsf{E}_\to$, 5, 1
7.	Rab	$\mathsf{E}_\forall$, 6

Exercise 7 (10 points)**.** Consider the following properties of relations.

- Transitivity (TR),
- Antisymmetry (AS),
- Irreflexivity (IR).

Exactly one of these properties follows logically from the other two combined. This means that if a relation has two of these properties, it has the third one as well.

Which of the properties follows from the other two? Motivate your answer with an argument.

Solution. We argue that if a relation is transitive and irreflexive, then it must be antisymmetric.

Suppose a relation R is transitive and irreflexive. Assume that we have Rxy and Ryx. By transitivity this means that Rxx which is impossible give irreflexivity. So we can never have Rxy and Ryx.