Logic and conversation: the case of free choice

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Abstract

Free choice inferences represent a much discussed case of a divergence between logic and language (Kamp 1973, Zimmermann 2000). Grice influentially argued that the assumption that such divergence does in fact exist is a mistake originating "from inadequate attention to the nature and importance of the conditions governing conversation" (Grice 1989: 24). I will first show that when applied to free choice phenomena, the standard implementation of Grices view, modelling semantics and pragmatics as two separate components, is empirically inadequate. I will then propose a different account: a bilateral state-based modal logic modelling next to literal meanings also pragmatic factors and the additional enriched meanings that arise from their interaction. The pragmatic factor I will consider connects to a tendency of language users to neglect empty configurations when engaging in linguistic interpretation. The non-emptiness atom (NE) from team semantics provides a perspicuous way to formally represent this tendency and to rigorously study its impact on interpretation. In terms of NE, I will define a pragmatic enrichment function and show that, in interaction with disjunction occurring in positive contexts and only in these cases, pragmatic enrichment yields non-trivial effects including predicting free choice inferences and their cancellation under negation. The latter result relies on the adopted bilateralism, where each connective comes with an assertion and a rejection condition and negation is defined in terms of the latter notion.

1 Introduction

The relation between literal meaning (semantics) and inferences based on language use (pragmatics) has been the subject of a longstanding debate in philosophy and linguistics and important progress has been made in the development of diagnostics to distinguish between semantic and pragmatic inference and in the formal derivation of the latter from general principles of

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		pragm. derivable	cancel lable	non-embed.	proc. cost	acqui sition
Pra gma tics	Conversational implicature J is always very punctual ⇝ J is not a good philosopher	+	+	+	high	late
Sem ant ics	Classical entailment I read some novels ⇝ I read something	−	−	−	low	early
3rd Kind	Epistemic Indefinites *Irgendjemand* hat angerufen ⇝ Speaker doesn't know who	+	−	+	?	?
	Modified Numerals Al has *at least two* degrees ⇝ Maybe two, maybe more	+	−	+	?	?
	FC disjunction You may do A *or* B ⇝ You may do *A*	+	?	?	low	early
	Scalar implicature I read some novels ⇝ I didn't read all novels	+	+	?	high	late

Table 1: Beyond Gricean paradise

conversation.[1] On the canonical view, pragmatic inference (aka conversational implicature) is derivable by general conversational principles, is cancellable, and is not embeddable under logical operators. Semantic inference, instead, lacks all these properties.

A number of inferences have been recently discussed in the literature which challenge the semantics vs pragmatics divide emerging from the canonical view. These include *ignorance inferences* in epistemic indefinites[2] and modified numerals[3] and phenomena of *free choice* (FC), where conjunctive meanings are unexpectedly derived for disjunctive sentences.[4] The common core of these inferences is that although they are derivable by conversational principles they lack at least one of the other defining properties of canonical pragmatic inference: they are often non-cancellable, they are sometimes embeddable, they are acquired early and their processing time has been shown to equal that of literal interpretations. See Table 1 for illustrations.

The overall goal of this project is to arrive at a formal account of these inferences which captures their quasi-semantic behaviour while explaining their pragmatic nature. The general strategy is to develop *logics of conversation*, operating at the level of speech acts, which model

[1] Grice (1975, 1989); Gazdar (1979); Horn (1984); Levinson (1983, 2000); Sperber and Wilson (1995) and many others afterwards.

[2] E.g., Jayez and Tovena (2006); Alonso-Ovalle and Menéndez-Benito (2010, 2015); Aloni and Port (2010, 2015).

[3] E.g., Geurts and Nouwen (2007); Coppock and Brochhagen (2013); Schwarz (2016); Ciardelli *et al.* (2018a); van Ormondt (2019).

[4] Kamp (1973); Simons (2000); Zimmermann (2000); Geurts (2005); Klinedinst (2006); Aloni (2007); Fox (2007); Barker (2010) and many others.

next to literal meanings also pragmatic factors and the additional inferences that arise from their interaction.

The present article focuses on the case of FC disjunction. In FC inferences, conjunctive meanings are derived from disjunctive modal sentences contrary to the prescriptions of classical logic:[5]

(1) $\Diamond(\alpha \vee \beta) \rightsquigarrow \Diamond\alpha \wedge \Diamond\beta$

Examples (2) and (3) illustrate cases of FC inference involving a deontic and an epistemic modal verb:

(2) Deontic FC (Kamp, 1973)
  a. You may go to the beach or to the cinema.
  b. $\rightsquigarrow$ You may go to the beach and you may go to the cinema.

(3) Epistemic FC (Zimmermann, 2000)
  a. Mr. X might be in Victoria or in Brixton.
  b. $\rightsquigarrow$ Mr. X might be in Victoria and he might be in Brixton.

Influential existing accounts of FC inferences view them as closely related to scalar implicatures (e.g., Fox, 2007; Chemla, 2008; Chierchia *et al.*, 2011; Franke, 2011; Bar-Lev and Fox, 2020). FC inferences and scalar implicatures, however, are different phenomena. As indicated in Table 1, they differ in their processing cost (Chemla and Bott, 2014) and in how early they are acquired (Tieu *et al.*, 2016). They further display different cancellability potential:

(4) I read some novels, in fact I read them all. (scalar)

(5) You may go to Paris or Rome, ?? in fact you may not go to Paris. (FC)

And, finally, in contrast to scalar implicatures, FC inferences appear more to be part of *what is said* than of *what is merely implicated*, as shown by the following examples (modified from Mastop, 2005; Aloni, 2007):

(6) MOTHER: You may do your homework or help your father in the kitchen.
SON GOES TO THE KITCHEN.
FATHER: Go to your room and do your homework!
SON: But, mom said I could also help you in the kitchen.

(7) MOTHER: Al is French or Italian.
FATHER: She is both.
SON: ??But, mom said she is not both.

In view of these differences we conclude that FC inferences are not special cases of scalar implicatures. While scalar implicatures are naturally derived by applications of (grammaticalised versions of) Grice's Quantity Maxim and typically rely on a comparison with a relevant set of (lexical) alternatives, in our account alternatives will play no role and the operative pragmatic factor will be a tendency of language users which at most connects to a version of Grice's Maxim of Quality.

The novel hypothesis at the core of this proposal is that FC and related inferences are neither the result of conversational reasoning (as in traditional gricean and neo-gricean approaches, e.g., Sauerland, 2004; Simons, 2010), nor the effect of spontaneous optional applications of grammatical operators (as in the grammatical view of scalar implicatures, Chierchia *et al.*, 2011).

[5] FC interpretations are compatible with exclusive readings of disjunction, which would lead to the additional inference $\neg\Diamond(\alpha \wedge \beta)$. In fact $\Diamond\alpha \wedge \Diamond\beta \neq \Diamond(\alpha \wedge \beta)$.

Rather they are a straightforward consequence of something else speakers do in conversation, namely, when interpreting a sentence they create structures representing reality, pictures of the world (Johnson-Laird, 1983) and in doing so they systematically neglect structures which verify the sentence by virtue of some empty configuration. This tendency, which I will call *neglect-zero*, follows from the expected difficulty of the cognitive operation of evaluating truths with respect to empty witness sets (Bott *et al.*, 2019). Models which verify a sentence by virtue of some empty set will be called *zero-models*. As an illustration consider the following examples:

(8) Every square is black.

a. Verifier: [■, ■, ■]
b. Falsifier: [■, □, ■]
c. Zero-models: [ ]; [△, △, △]; [◇, ▲, ♠]

(9) Less than three squares are black.

a. Verifier: [■, □, ■]
b. Falsifier: [■, ■, ■]
c. Zero-models: [ ]; [△, △, △]; [◇, ▲, ♠]

The interpretation of (8) and (9) leads to the creation of structures representing reality, some verifying the sentence (the models depicted in (a)), some falsifying it (the models in (b)). The neglect-zero hypothesis states that zero-models, the ones represented in (c), are usually kept out of consideration. Zero-models are neglected because they are cognitively taxing. Findings from number cognition confirm this difficulty, which also explains the special status of the zero among the natural numbers (Nieder, 2016)[6]; the existential import effects operative in the logic of Aristotle (the inference from *every A is B* to *some A is B*); and why downward-monotonic quantifiers are more difficult to process than upward-monotonic ones (Bott *et al.*, 2019). Encoding the absence of objects rather than their presence empty witnesses are more detached from sensory experience and, therefore, harder to conceive. The inference from the perception of absence to the truth of a sentence brings in additional costs, which results in a systematic dispreference for zero-models, a neglect-zero tendency. The idea at the core of my proposal is that FC and related inferences, just like the Aristotelian existential import effects, are a consequence of such neglect-zero tendency assumed to be operative among language users in ordinary conversations.

The next section introduces the phenomenon of FC and the challenges it involves for logic-based accounts of linguistic meaning; Section 3 presents the core idea of the proposal introducing the main ingredients for a formalisation of neglect-zero effects on our interpretation of disjunctive sentences using tools from team semantics; Section 4 defines the formal system, a bilateral version of a team-based modal logic called Bilateral State-based Modal Logic (BSML); Sections 5 and 6 present the main results and applications; Section 7 compares different implementations of neglect-zero effects in variants of BSML and discusses how they can be used to model different interpretation strategies speakers can adopt in conversation; Section 8 compares with a previous approach and Section 9 concludes.[7]

2 The paradox of free choice

As mentioned in the introduction, sentences of the form "You may A or B" are normally understood as implying "You may A and you may B". The following, however, is not a valid principle in standard deontic logic (von Wright, 1968).

[6]Its late invention in human history, its late emergence in human development, its limited evolution in animals, and its special representation in the brain (Nieder, 2016).

[7]Section 2 and parts of sections 4 and 6 are largely based on previous unpublished work by the author.

(10) $\Diamond(\alpha \vee \beta) \rightarrow \Diamond\alpha$ [Free Choice (FC) Principle]

As Kamp (1973) pointed out, plainly making the FC principle valid, for example by adding it as an axiom, would not do because it would allow us to derive any $\Diamond b$ from $\Diamond a$ as shown in (11):

(11)
1. $\Diamond a$ [assumption]
2. $\Diamond(a \vee b)$ [from 1, by classical reasoning]
3. $\Diamond b$ [from 2, by FC principle]

The step leading to 2 in the derivation above uses the following classically valid principle derivable from the introduction rule of disjunction, also known as addition, and the upward monotonicity of $\Diamond$:

(12) $\Diamond\alpha \rightarrow \Diamond(\alpha \vee \beta)$ [Modal Addition]

In natural language, however, (12) seems invalid: *You may post this letter* doesn't seem to imply *You may post this letter or burn it*, while (10) seems to hold, in direct opposition to the principles of deontic logic. (von Wright, 1968, chapter I, section 7) called this the paradox of free choice permission. Related paradoxes arise also for imperatives (see Ross' (1941) paradox), and other modal constructions.

Several solutions have been proposed to the paradox of free choice permission. Many have argued that what we called the FC principle is not a logical validity but a conversational implicature, a pragmatic inference derived as the product of rational interactions between cooperative language users rather than logic. The step leading to 3 in derivation (11) is therefore unjustified. Various ways of deriving free choice inferences as conversational implicatures have been proposed (e.g., Gazdar 1979, Kratzer and Shimoyama 2002, Schulz 2005 and Franke, 2011).

One argument in favour of a pragmatic account of FC comes from the observation that free choice effects disappear in negative contexts. For example, sentence (13) cannot merely mean that you cannot choose between the cake and the ice-cream as one would expect if free choice effects were semantic entailments rather than pragmatic implicatures (Alonso-Ovalle, 2006):

(13) Dual Prohibition

a. You are not allowed to eat the cake or the ice cream.
$\rightsquigarrow$ You are not allowed to eat either one.

b. $\neg\Diamond(\alpha \vee \beta) \rightsquigarrow \neg\Diamond\alpha \wedge \neg\Diamond\beta$

Others have proposed non-classical modal logic systems where the step leading to 3 in (11) is justified, but either the transitivity of the consequence relation fails, as in Goldstein (2019), or the step leading to 2 is no longer valid, as for example in Aloni (2007), who proposed a uniform semantic account of free choice effects of disjunctions and indefinites under both modals and imperatives.[8]

One argument in favor of a semantic approach comes from the observation that FC effects can embed under some operators, for example under universal quantifiers, as experimentally attested by Chemla (2009):

(14) Universal FC

a. All of the boys may go to the beach or to the cinema.
$\rightsquigarrow$ All of the boys may go to the beach and all of the boys may go to the cinema.

b. $\forall x\Diamond(\alpha \vee \beta) \rightsquigarrow \forall x(\Diamond\alpha \wedge \Diamond\beta)$

[8] Simons (2005) and Barker (2010) also proposed semantic accounts of free choice inferences, the latter crucially employing an analysis of *or* in terms of linear logic additive disjunction combined with a representation of strong permission using the deontic reduction strategy as in Lokhorst (2006). While Aloni (2007) and Simons (2005) fail to account for Dual Prohibition cases, Barker (2010) derives it as a pragmatic inference.

Actually, examples like (14) are often used to argue against globalist accounts of implicatures, rather than in favor of semantic accounts. Notice however that localist accounts (Fox, 2007; Chierchia *et al.*, 2011) who predict the availability of embedded FC implicatures and therefore capture (14), need adjustments to capture the Dual Prohibition case illustrated in (13).[9]

A third argument which can be used against most approaches mentioned so far comes from the observation that FC effects also arise with configurations where disjunction takes wide scope with respect to the modal (Zimmermann, 2000):

(15) Wide Scope FC
  a. Detectives may go by bus or they may go by boat. $\rightsquigarrow$ Detectives may go by bus and may go by boat.
  b. Mr. X might be in Victoria or he might be in Brixton. $\rightsquigarrow$ Mr. X might be in Victoria and might be in Brixton.
  c. $\Diamond\alpha \vee \Diamond\beta \rightsquigarrow \Diamond\alpha \wedge \Diamond\beta$

Wide scope FC inferences are hard to derive by Gricean means. Most pragmatic analyses of FC indeed only derive the narrow scope case (exceptions are Schulz (2005) and Chemla (2008)) and attempt to reduce all surface wide scope FC examples to cases of narrow scope FC:

(16) a. Detectives may go by bus or they may go by boat.
  $\rightsquigarrow$ Detectives may go by bus
  b. Logical Form: $\Diamond(\alpha \vee \beta)$ [rather than $\Diamond\alpha \vee \Diamond\beta$]

One argument against such a reductive strategy has been presented by Alonso-Ovalle (2006). Example (17) gives rise to a free choice inference, but an analysis of (17) as a narrow scope disjunction would require dubious syntactic operations (e.g., Simons' (2005) covert across-the-board (ATB) movement of the modal would not work here, because ATB movement requires identical modals in each clause):

(17) a. You may email us or you can reach the Business License office at 949 644-3141. $\rightsquigarrow$ You may email us
  b. Logical Form: $\Diamond\alpha \vee \Diamond\beta$ [it cannot be $\Diamond(\alpha \vee \beta)$]

One account which captures wide scope FC inference is the one defended by Zimmermann (2000) and further refined by Geurts (2005). Rather than attempting to develop an alternative to classical logic, Zimmermann (2000) criticises the standard logical translation of natural language *or* into Boolean disjunction $\vee$, and proposes instead a modal analysis of linguistic disjunction, which, as (18) illustrates, should be treated as a conjunctive list of epistemic possibilities (rather than a Boolean disjunction). In (18) $\Diamond$ must be read as an epistemic possibility operator:

(18) A or $B \mapsto \Diamond\alpha \wedge \Diamond\beta$

Zimmermann then distinguishes between (15-c), which, according to him, is an unjustified logical principle, and the following intuitively valid natural language principle:

(19) X may A or may B $\rightsquigarrow$ X may A and X may B

By analysing disjunctions as conjunctions of epistemic possibilities, as in (18), Zimmermann argues that the correct logical rendering of (19) is (20), which, if derived, explains our wide scope FC intuitions (again $\Diamond$ is an epistemic possibility operator and P is a deontic possibility

[9] Localists usually explain FC cancellation under negation by appealing to variations of the Strongest Meaning Hypothesis (Dalrymple *et al.*, 1998). Notice that also semantic accounts can adopt the same strategy (Aloni, 2007, page 81).

	NS FC	Dual Prohibition	Universal FC	WS FC
Semantic	yes	**no**	yes	?
Pragmatic	yes	yes	**no**	?

Table 2: Semantic and pragmatic accounts of FC.

operator):[10]

(20) $(\Diamond P\alpha \wedge \Diamond P\beta) \rightarrow (P\alpha \wedge P\beta)$

The account defended in this article shares the basic intuition of Zimmermann's and Geurts' analyses, namely that when one utters A or B, one normally conveys that each disjunct is an open epistemic possibility. My implementation of this idea, however, will be rather different from Zimmermann's or Geurts' and, as I hope to be able to show, it will be better equipped to capture the complex range of facts FC sentences give rise to when embedded under other logical operators.

To summarize, we have observed the following facts evidencing a hybrid behaviour of FC inferences in between semantics and pragmatics:

(21) a. $\Diamond(\alpha \vee \beta) \rightsquigarrow \Diamond\alpha \wedge \Diamond\beta$ [Narrow Scope (NS) FC]
b. $\neg\Diamond(\alpha \vee \beta) \rightsquigarrow \neg\Diamond\alpha \wedge \neg\Diamond\beta$ [Dual Prohibition]
c. $\forall x\Diamond(\alpha \vee \beta) \rightsquigarrow \forall x(\Diamond\alpha \wedge \Diamond\beta)$ [Universal FC]
d. $\Diamond\alpha \vee \Diamond\beta \rightsquigarrow \Diamond\alpha \wedge \Diamond\beta$ [Wide Scope (WS) FC]

Several accounts have been proposed, which capture Narrow Scope FC or Wide Scope FC, but as summarized in Table 2, a purely semantic or pragmatic analysis typically fails to scale up to capture the whole range of relevant facts (at least without the help of additional assumptions, such as the Strongest Meaning Hypothesis or others).[11]

In the following sections I will present a logic-based account of FC inference beyond the canonical semantics *vs* pragmatics divide. The standard implementation of the Gricean view assumes two separate components: semantics, on one side, ruled by classical logic; and pragmatics, on the other side, ruled by general principles of conversation. I propose a different account: I develop a non-classical (state-based) modal logic where (i) besides literal meanings also pragmatic factors can be modelled which can intrude in the process of interpretation and (ii) an operation of pragmatic enrichment (denoted by a function $[\ ]^+$) can be defined in terms of such intrusion. The resulting system derives FC and related inferences, but only for pragmatically enriched formulas:

$$[\Diamond(\alpha \vee \beta)]^+ \models \Diamond\alpha \wedge \Diamond\beta, \text{ but } \Diamond(\alpha \vee \beta) \not\models \Diamond\alpha \wedge \Diamond\beta$$

[10]Zimmermann actually only derives the weaker principle in (i) (under certain assumptions including his Authority principle):

(i) $(\Diamond P\alpha \wedge \Diamond P\beta) \rightarrow (\Box P\alpha \wedge \Box P\beta)$

$\Box\alpha$ should be read here as "it is certain that α". Geurts' (2005) refinement avoids this problem.

[11]A number of recent semantic or localist accounts do much better than what is summarised in Table 2. For example, Aher (2012) and Willer (2018) capture Dual Prohibition facts; Bar-Lev and Fox (2020) discuss strategies to account for both Dual Prohibition and Wide Scope FC in a localist framework; Hawke and Steinert-Threlkeld (2018) account for Wide Scope FC (but only of the epistemic kind); Starr (2016) accounts for both Dual Prohibition and Wide Scope FC (but only of the deontic kind) and, finally, Goldstein (2019) for both Dual Prohibition and Wide Scope FC of the epistemic and deontic kind. Goldstein's dynamic homogeneity approach makes predictions which are very close to mine. I will comment on some of the differences in Section 8.

The upshot of our logic-based account is that the whole range of hybrid behaviour summarised in (21) can be naturally derived, but, again, only for pragmatically enriched formulas. Literal meanings (denoted by $[\ ]^+$-free formulas) maintain their classical behaviour. The paradoxical reasoning in (11) is predicted to fail as a case of equivocation:

(22)

1.	$\Diamond\alpha$		
2.	$\Diamond(\alpha \vee \beta)$	$\neq$	$[\Diamond(\alpha \vee \beta)]^+$
3.			$\Diamond\beta$

As mentioned in the introduction the relevant pragmatic factor I will consider connects to a tendency of language users to neglect zero-models when engaging in linguistic interpretations. As we will see in the next section, the non-emptiness atom (NE) from team semantics (Väänänen, 2007; Yang and Väänänen, 2017) provides a perspicuous way to formally represent this tendency and to rigorously study its impact on interpretation.

3 The logic of conversation

The development of a logic deriving FC inferences is not a trivial task. As mentioned in the previous section, Kamp (1973) observed that adding the FC principle ($\Diamond(\alpha \vee \beta) \rightarrow \Diamond\alpha$) to classical modal logic generates explosion allowing us to derive any $\Diamond\beta$ from any other $\Diamond\alpha$:

(23)

1.	$\Diamond\alpha$	[assumption]
2.	$\Diamond(\alpha \vee \beta)$	[from 1, by modal addition]
3.	$\Diamond\beta$	[from 2, by FC principle]

Goldstein (2019) further showed that a system validating both the FC principle and the principle of Dual Prohibition ($\neg\Diamond(\alpha \vee \beta) \rightarrow \neg\Diamond\alpha$) gives rise to equally paradoxical results:

(24)

1.	$\neg\Diamond\alpha$	[assumption]
2.	$\neg\Diamond(\alpha \vee \beta)$	[from 1, by FC principle and contraposition]
3.	$\neg\Diamond\beta$	[from 2, by dual prohibition]

On my view, the source of the problem highlighted by these two cases is the mistaken attempt to explain free choice facts purely in terms of truth-conditions. On my proposal FC inferences are not truth-conditional effects but rather a direct consequence of a conversational factor, what I have called a neglect-zero tendency, which will be modelled by a pragmatic enrichment function $[\ ]^+$. Inferences derived by such pragmatic enrichments do not relate propositions but rather speech acts and therefore might diverge from classical semantic entailments (see Stalnaker's (1975) notion of a *reasonable inference*). Whenever A is *true*, also (A or B) is true [ADDITION], but it does not follow that whenever A is *assertable*, (A or B) is equally assertable [FAILURE OF ADDITION]. Similarly, if whenever A is true, B is also true, then whenever B is *false*, A must also be false [CONTRAPOSITION], but B can be *rejectable* in conversation without A being rejectable even if whenever A is assertable B is also assertable [FAILURE OF CONTRAPOSITION]. To model such *conversational inferences* I will employ a bilateral version of team-based modal logic modelling assertion and rejection conditions rather than truth. Operating at the level of speech acts rather than truth-conditions our logic will diverge from classical logic (e.g., invalidate addition and contraposition), but only for pragmatically enriched formulas.

- Addition: $\alpha \models \alpha \vee \beta$, but $[\alpha]^+ \not\models [\alpha \vee \beta]^+$
- Contraposition: $\alpha \models \beta \;\Rightarrow\; \neg\beta \models \neg\alpha$, but $[\alpha]^+ \models [\beta]^+ \;\not\Rightarrow\; [\neg\beta]^+ \models [\neg\alpha]^+$

The team-basedness of the system will further be crucial to formalise the neglect-zero tendency at the core of our derivation of FC effects.

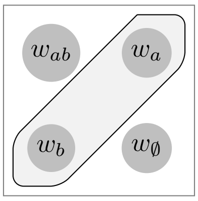


Figure 1: $M, s \not\models a$; $M, s \not\models \neg a$

3.1 Team semantics and bilateralism

In team semantics a formula is interpreted with respect to a set of points of evaluation (a team) rather than a single point. In team-based modal logic a team is a set of possible worlds. Let $M = \langle W, R, V \rangle$ be a classical Kripke model where W is a non-empty set of possible worlds, R an accessibility relation over W and V a world-dependent valuation function. In classical modal logic formulas are evaluated with respect to single possible worlds (elements of W); in team-based modal logic with respect to sets of possible worlds (subsets of W):

- Classical modal logic:
$$M, w \models \phi, \text{ where } w \in W$$
- Team-based modal logic:
$$M, t \models \phi, \text{ where } t \subseteq W$$

I will employ a *bilateral* version of a team-based modal logic, where teams are interpreted as *information states* and both support ($\models$) and anti-support ($\mathrel{=\!\!|}$) conditions are defined, meant to capture the assertability and rejectability of a sentence in a state:

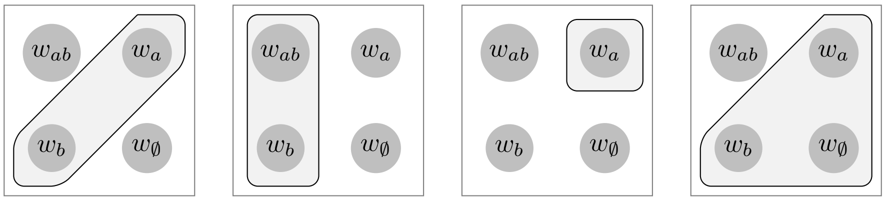
- Bilateral state-based modal logic:
$$M, s \models \phi, \text{ “}\phi\text{ is assertable in state } s\text{”}, \quad \text{with } s \subseteq W$$
$$M, s \mathrel{=\!\!|} \phi, \text{ “}\phi\text{ is rejectable in state } s\text{”}, \quad \text{with } s \subseteq W$$

Classical modal logic models *truth* in a possible world. Bilateral state-based modal logic models *assertion* and *rejection* conditions in an information state (a set of possible worlds).

In a state-based system, while logical consequence can be classical, we can still have states where neither ϕ nor $\neg\phi$ are supported, and, therefore, just like in supervaluationism (van Fraassen, 1966), we can reject bivalence while validating the Law of Excluded Middle.[12] Information states (teams) are then less determinate entities than possible worlds comparable to truthmakers (van Fraassen, 1969; Fine, 2017), possibilities (Humberstone, 1981; Holliday, 2015), or situations (Barwise and Perry, 1983).[13] The partial nature of an information state makes state-based systems particular suitable for capturing phenomena at the semantics-pragmatics

[12]See Figure 1 for an illustration of a state which neither supports a nor $\neg a$. In the picture, w_a stands for a world where only a is true, w_b only b, etc. These four possible worlds will be used for illustration throughout the paper.

[13]In truthmaker semantics and possibility semantics formulas are interpreted with respect to points in a partially ordered set (as in a Kripke semantics for intuitionistic logic), rather than with respect to sets of points of evaluation (teams). But the powerset of any set is isomorphic to a complete atomic Boolean lattice so all these systems are closely related. One of the advantages of defining information states as elements of a partially ordered set rather than as a team is that one may want to include partially ordered sets not satisfying the properties of a Boolean lattice. There are two main reasons for my choice to characterise states as teams nevertheless: (i) I will mainly focus on linguistic applications and I have not yet found any compelling linguistic evidence favouring the more abstract algebraic characterisation; (ii) this material deals with disjunction and the notion of disjunction I adopt is more perspicuously defined if we characterise states as sets rather than points, as in team semantics.



(a) $\models (a \vee b)$; and $\models [a \vee b]^+$ (b) $\models (a \vee b)$; and $\models [a \vee b]^+$ (c) $\models (a \vee b)$; but $\not\models [a \vee b]^+$ (d) $\not\models (a \vee b)$; and $\not\models [a \vee b]^+$

Figure 2: Comparison $a \vee b$ and $[a \vee b]^+$

interface, including anaphora (Groenendijk and Stokhof, 1991; Groenendijk *et al.*, 1996; Dekker, 2012), questions (Ciardelli and Roelofsen, 2011; Ciardelli *et al.*, 2018b), epistemic modals (Veltman, 1996). The present article focuses on phenomena of FC. Crucial for this application is the interpretation of disjunction as tensor or split disjunction (Cresswell, 2004; Väänänen, 2007; Yang and Väänänen, 2017; Hawke and Steinert-Threlkeld, 2018).[14]

3.2 Split Disjunction

A state s supports a split disjunction $\phi \vee \psi$ iff s is the union of (can be split into) two substates, each supporting one of the disjuncts:[15][16]

$$M, s \models \phi \vee \psi \quad \text{iff} \quad \exists t, t' : t \cup t' = s \;\&\; M, t \models \phi \;\&\; M, t' \models \psi$$

As an illustration consider the states represented in Figure 2. In these pictures again w_a stands for a world where only a is true, w_b only b, etc. The split disjunction $(a \vee b)$ is supported by the first three states, but not by 2(d). The latter contains $w_\emptyset$, a world where both a and b are false. No state containing such a world can be the union of two substates, each one supporting one of a and b. The state represented in 2(c), instead, does support $(a \vee b)$. This is because

[14]As far as I know, Hawke and Steinert-Threlkeld (2018) were the first to use split disjunction to explain FC phenomena. Their account however only applies to the case of epistemic modals, and, at least in the 2018 version, only captured the case of wide scope FC.

[15]There are at least two other ways to define disjunction in a state-based semantics (see Aloni, 2018, for a detailed comparison):

$$M, s \models \phi \vee_A \psi \quad \text{iff} \quad M, s \models \phi \text{ or } M, s \models \psi$$
$$M, s \models \phi \vee_B \psi \quad \text{iff} \quad \forall w \in s : M, \{w\} \models \phi \text{ or } M, \{w\} \models \psi$$

Split disjunction is the canonical way in team semantics because (i) it yields classical logic in combination with classical formulas, which is not the case for the inquisitive disjunction $\vee_A$ which leads to violations of the Law of Excluded Middle even in combination with classical disjuncts; and (ii) it correctly interacts with formulas which are not downward-closed, which does not hold for $\vee_B$, the notion usually adopted in possibility semantics and dynamic semantics. All three notions collapse if s is a singleton, which corresponds to the classical case.

[16]An anonymous reviewer asked for linguistic motivations for the adoption of split disjunction besides free choice and related inferences (ignorance and distribution, see section 6.2). As said, split disjunction is the canonical way to define disjunction in team semantics because it delivers classical disjunction when combined with classical formulas (i.e. formulas without NE). Therefore any linguistic evidence justifying the adoption of classical disjunction as the meaning of 'or' also justifies the adoption of split disjunction in a state-based setting and the adoption of a state-based setting is justified by the pragmatic nature of the phenomena addressed, which arguably concern inferences relating speech-acts rather than propositions. Furthermore, split disjunction interacts correctly with non classical formulas which are not downwards-closed, some of which have potential for interesting linguistic applications like the inconstancy or non-dependence atoms (Galliani, 2015), which can be used to model variation requirements operative in some marked indefinites (e.g., epistemic and non-specific indefinites, see Aloni and Degano, 2022).

(i) according to the definition above, the substates necessary for verifying the disjunction can be empty, and (ii) in our logic the empty state supports all classical propositions. Therefore whenever a state s supports one classical disjunct, in this case a, we can find suitable substates of s supporting each classical disjunct: the state itself and the empty state.

The state represented in Figure 2(c) is an example of what in the introduction I called a *zero-model* for $(a \vee b)$, i.e. a model which verifies the formula by virtue of an empty witness. In the next section I will define a notion of pragmatic enrichment whose core effect is to disallow such zero-models. When applied to a disjunction pragmatic enrichment will crucially rule out the possibility of the empty state acting as one of the relevant substates for its evaluation. As we will see, a state s supports a pragmatically enriched disjunction $[\alpha \vee \beta]^+$ iff s is the union of two *non-empty* substates, each supporting one of the disjuncts. Pragmatically enriched disjunctions thus require both their disjuncts to be live possibilities, as in Zimmermann (2000) and Geurts (2005). Formula $[a \vee b]^+$ is no longer supported by the state represented in 2(c) because none of the worlds in the state verifies b (b is not a live possibility), and so no non-empty substate of the state can be found which supports the second disjunct. The pragmatic enrichment function $[\ ]^+$ is defined in terms of the so-called Non-Emptiness atom NE also from team semantics (Väänänen, 2007; Yang and Väänänen, 2017).

3.3 Non-emptiness and pragmatic enrichment

At the core of my proposal is the assumption that language users tend to neglect zero-models because of their cognitive costs and that FC and related inferences are a direct consequence of this behaviour. The non-emptiness atom (NE) from team semantics provides a perspicuous way to formally represent this tendency and to rigorously study its impact on interpretation. NE is supported in a state if and only if the state is not empty.

$$M, s \models \text{NE} \quad \text{iff} \quad s \neq \emptyset$$

By enriching formulas with the requirement to satisfy NE distributed along each of their subformulas, we can rule out the possibility of verifying positive sentences by virtue of empty witnesses, in fact modelling the neglect-zero tendency at the level of propositions. This idea is formally implemented by the pragmatic enrichment function $[\ ]^+$, defined as follows (henceforth I will use α, β as metavariables ranging over formulas in the NE-free fragment of the language):

$$\begin{aligned}
[p]^+ &= p \wedge \text{NE} \\
[\neg\alpha]^+ &= \neg[\alpha]^+ \wedge \text{NE} \\
[\alpha \vee \beta]^+ &= ([\alpha]^+ \vee [\beta]^+) \wedge \text{NE} \\
[\alpha \wedge \beta]^+ &= ([\alpha]^+ \wedge [\beta]^+) \wedge \text{NE} \\
[\Diamond\alpha]^+ &= \Diamond[\alpha]^+ \wedge \text{NE}
\end{aligned}$$

As we will see, the unrestricted application of pragmatic enrichment derives narrow scope and wide scope FC inferences (the latter with restrictions) while no undesirable side effects obtain in other configurations, in particular under negation:

- Narrow scope FC: $[\Diamond(\alpha \vee \beta)]^+ \models \Diamond\alpha \wedge \Diamond\beta$
- Wide scope FC: $[\Diamond\alpha \vee \Diamond\beta]^+ \models \Diamond\alpha \wedge \Diamond\beta$ (with restrictions)
- Dual Prohibition: $[\neg\Diamond(\alpha \vee \beta)]^+ \models \neg\Diamond\alpha \wedge \neg\Diamond\beta$

The FC results rely on the adoption a notion of **modality** similar to the one employed in possibility semantics (Humberstone, 1981; Holliday, 2015), and inquisitive modal logic (Ciardelli, 2016)[17]. A state s supports a possibility modal sentence $\Diamond\phi$ iff for all worlds in s there is a non-empty subset of the set of worlds accessible from w which supports ϕ.

$$M, s \models \Diamond\phi \quad \text{iff} \quad \forall w \in s : \exists t \subseteq R[w] : t \neq \emptyset \ \& \ M, t \models \phi$$

The Dual Prohibition result relies on the adopted bilateralism, where each connective comes with an assertion ($\models$) and a rejection ($=\!\!|$) condition and **negation** is defined in terms of the latter notion (see Aher, 2012; Willer, 2018, who also adopt a bilateral notion of negation to capture Dual Prohibition):

$$\begin{array}{lll} M, s \models \neg\phi & \text{iff} & M, s =\!\!| \ \phi \\ M, s =\!\!| \ \neg\phi & \text{iff} & M, s \models \phi \end{array}$$

The analysis resulting from our state-based account is not only empirically correct (e.g., some of the predictions have been experimentally confirmed, Cremers *et al.*, 2017; Tieu *et al.*, 2019), but also cognitively plausible. Our account shows that FC inferences follow from the assumption that in ordinary conversations language users tend to neglect zero-models because of their cognitive cost. It is an empirical question whether language users do in fact behave as assumed but, if they do, FC inferences would follow at cost zero, explaining so their low processing time (Chemla and Bott, 2014) and their early acquisition (Tieu *et al.*, 2016), both unexpected facts on an account which views FC inferences as a kind of scalar implicatures.

Our analysis further highlights a connection between the cognitive cost of representing the empty set and the observed systematic compliance of language users with a principle which prescribes to avoid contradictions ('avoid $\bot$').[18] As stated above, in a state-based semantics the empty state vacuously supports every classical formula, including contradictions: $M, \emptyset \models p \wedge \neg p$. We may call $\emptyset$ the state of logical absurdity. Given this characterisation, while in classical logic there is no non-trivial way to model 'avoid $\bot$' (after all, $\neg\bot = \top$), in a state-based system we can model 'avoid $\bot$' as prescribing to avoid the state of logical absurdity, or, equivalently, as requiring the supporting state to be *non-empty.*[19] This is precisely the interpretation of the constant NE.[20] On this view the systematic compliance of speakers to the principle on non-contradiction is closely connected to a general preference in human cognition for the concrete (a non-empty state) over the abstract (the empty set) (Paivio, 1965; Mkrtychian *et al.*, 2019, and many others).

The next section formally introduces Bilateral State-based Modal Logic (BSML).

4 Bilateral State-based Modal Logic

Our target language L is the language of propositional modal logic enriched with the non-emptiness atom NE which as we said will be used to define the pragmatic enrichment function $[\ ]^+$.

Definition 1 (Language) *Let A be a set of sentential atoms $A = \{p, q, \dots\}$.*

$$\phi \quad := \quad p \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \Diamond\phi \mid \text{NE}$$

where $p \in A$.

[17]But different from the notion of modality employed in modal dependence logic (Väänänen, 2008). See (Anttila, 2021) for a comparison.

[18]'Avoid $\bot$' follows from Grice's maxim of QUALITY ('make your contribution one that is true').

[19]This connects to the idea behind the "inquisitive sincerity" and "attentive sincerity" maxims that have been discussed by Roelofsen (2013); Westera (2013); Ciardelli *et al.* (2014).

[20]As we will see in section 5, in BSML we can define different sorts of contradictions: classical contradiction like $p \wedge \neg p$, also referred to as weak contradiction, and a strong contradiction, $\text{NE} \wedge \neg\text{NE}$. Although $p \wedge \neg p$ and $\text{NE} \wedge \neg\text{NE}$ are not logically equivalent, NE expresses the negation of both in the following sense: $\neg\text{NE} \equiv (p \wedge \neg p)$ and $\text{NE} \equiv \neg(\text{NE} \wedge \neg\text{NE})$. Note, however, that $\text{NE} \not\equiv \neg(p \wedge \neg p)$. More on this in section 5.

A Kripke model for L is a triple $M = \langle W, R, V\rangle$, where W is a set of worlds, R is an accessibility relation on W and V is a world-dependent valuation function for A.

Formulas in our language are interpreted in models M with respect to a state $s \subseteq W$. Both support, $\models$, and anti-support , $\dashv$, conditions are specified. $R[w]$ refers to the set $\{v \in W \mid wRv\}$.

Definition 2 (Semantic clauses)

$$
\begin{array}{rll}
M, s \models p & \textit{iff} & \forall w \in s : V(w, p) = 1 \\
M, s \dashv p & \textit{iff} & \forall w \in s : V(w, p) = 0 \\[4pt]
M, s \models \neg\phi & \textit{iff} & M, s \dashv \phi \\
M, s \dashv \neg\phi & \textit{iff} & M, s \models \phi \\[4pt]
M, s \models \phi \vee \psi & \textit{iff} & \exists t, t' : t \cup t' = s \ \&\ M, t \models \phi \ \&\ M, t' \models \psi \\
M, s \dashv \phi \vee \psi & \textit{iff} & M, s \dashv \phi \ \&\ M, s \dashv \psi \\[4pt]
M, s \models \phi \wedge \psi & \textit{iff} & M, s \models \phi \ \&\ M, s \models \psi \\
M, s \dashv \phi \wedge \psi & \textit{iff} & \exists t, t' : t \cup t' = s \ \&\ M, t \dashv \phi \ \&\ M, t' \dashv \psi \\[4pt]
M, s \models \Diamond\phi & \textit{iff} & \forall w \in s : \exists t \subseteq R[w] : t \neq \emptyset \ \&\ M, t \models \phi \\
M, s \dashv \Diamond\phi & \textit{iff} & \forall w \in s : M, R[w] \dashv \phi \\[4pt]
M, s \models \text{NE} & \textit{iff} & s \neq \emptyset \\
M, s \dashv \text{NE} & \textit{iff} & s = \emptyset
\end{array}
$$

On the intended interpretation $M, s \models \phi$ stands for 'formula ϕ is assertable in s' and $M, s \dashv \phi$ stands for 'formula ϕ is rejectable in s', where s stands for the information state of the relevant speaker.

A state s supports an atomic proposition p iff p is true in all worlds in s; and anti-supports p iff p is false in all worlds in s. A state s supports a negation $\neg\phi$ iff it anti-supports ϕ; and anti-supports $\neg\phi$ iff it supports ϕ. A state s supports a disjunction $\phi \vee \psi$ iff s is the union of two substates, each supporting one of the disjuncts; and anti-supports $\phi \vee \psi$ iff s anti-supports ϕ and anti-supports ψ. A state s supports a conjunction $\phi \wedge \psi$ iff s supports ϕ and supports ψ; and anti-supports $\phi \vee \psi$ iff s is the union of two substates, each anti-supporting one of the disjuncts. A state s supports $\Diamond\phi$ iff for all $w \in s$: there is a non-empty subset of the set of worlds accessible from w which supports ϕ; and anti-supports $\Diamond\phi$ iff for all $w \in s$: the set of worlds accessible from w anti-supports ϕ. And finally a state s supports NE iff s is not empty; and it anti-supports NE iff it is empty.

We adopt the following abbreviation: $\Box\phi := \neg\Diamond\neg\phi$, and therefore derive the following interpretation for the necessity modal:

$$
\begin{array}{rll}
M, s \models \Box\phi & \text{iff} & \text{for all } w \in s : R[w] \models \phi \\
M, s \dashv \Box\phi & \text{iff} & \text{for all } w \in s : \text{there is a } t \subseteq R[w] : t \neq \emptyset \ \&\ t \dashv \phi
\end{array}
$$

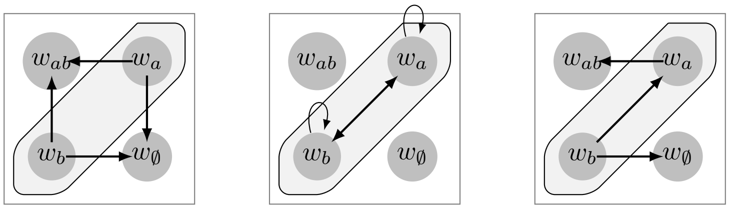
Logical consequence is defined as preservation of support.

Definition 3 (Logical consequence) $\phi \models \psi$ *iff for all* $M, s : M, s \models \phi \;\Rightarrow\; M, s \models \psi$

We also introduce a dependent notion of logical consequence, where we only consider model-state pairs (M, s) satisfying certain conditions.

Definition 4 (Logical consequence (restricted)) $\phi \models_X \psi$ *iff for all* $(M, s) \in X : M, s \models \phi \;\Rightarrow\; M, s \models \psi$

This restriction is used to express consequences which depend on "state-based" constraints on the accessibility relation R such as the ones defined in the following section.



(a) indisputable (but not state-based) (b) state-based (and so also indisputable) (c) neither

Figure 3: Indisputability vs state-basedness

4.1 State-based constraints on the accessibility relation

In a state-based modal logic we can formulate constraints on the accessibility relation in a model M which depend on a designated state in M. I will introduce two such constraints, indisputability and state-basedness, and propose to employ these constraints to capture differences between epistemic and deontic modal verbs.[21]

Definition 5 *Let* $M = (W, R, V)$ *and* $s \subseteq W$*.*

- *R is indisputable in* (M, s) *iff for all* $w, v \in s : R[w] = R[v]$
- *R is state-based in* (M, s) *iff for all* $w \in s : R[w] = s$

Indisputability relates to Zimmermann's (2000) Authority Principle. The property of being state-based relates to his Self-Reflection Principle.

An accessibility relation R is indisputable in a model-state pair (M, s) if any two worlds in s access exactly the same set of worlds according to R. Assuming s represents the information state of the relevant speaker, an indisputable R means that the speaker is fully informed about R, so, for example, if R represents a deontic accessibility relation, indisputability means that the speaker is fully informed about (or has full authority on) what propositions are obligatory or allowed.

An accessibility relation R is state-based in a model-state pair (M, s) if all and only worlds in s are R-accessible within s. Trivially if R is state-based, R is also indisputable. The adoption of a state-based R will lead to the satisfaction of the classical S5 axioms but also to an account of the infelicity of so-called epistemic contradictions, i.e. the assertion of the epistemic possibility of a proposition ϕ conjoined with its negation as in (25) (see Veltman, 1996; Yalcin, 2007; Hawke and Steinert-Threlkeld, 2018; Mandelkern, 2017, and others):

(25) #It might be raining but it is not raining.

The challenge for a logic-based account is to derive the infelicity or incoherence[22] of (25) ($\Diamond\phi \wedge \neg\phi \models \bot$), while preserving the non-factivity of $\Diamond$ ($\Diamond\phi \not\models \phi$). In BSML, we predict exactly this

[21] The properties defined in Definition 5 are not closed under bisimulation and therefore they are not modally definable. For this reason Anttila (2021) proposes a different definition for these notions and discusses different possible characterisations. This issue however has no impact on the linguistic applications discussed in this paper. Therefore we present here the simpler (not modally definable) definitions.

[22] Veltman (1996) and other dynamic systems predict a difference between (25), which is incoherent (no supporting state) but consistent (update does not necessarily lead to the state of absurdity) and the following variant, which is instead both incoherent and inconsistent:

(i) #It is not raining but it might be raining.

The distinction between coherence and consistency cannot be captured in the present static approach.

behaviour if we assume that the relevant accessibility relation is state-based (as illustrations consider $\Diamond a$, $\neg a$ and a interpreted in the model-state pair given in Figure 3(b)).

- Epistemic contradiction: $\Diamond\phi \wedge \neg\phi \models \bot$ [if R is state-based]
- Non-factivity: $\Diamond\phi \not\models \phi$ [even if R is state-based]

Epistemic contradiction seem to only arise with epistemic modals. The following sentences are coherent:

(26) You are here but you may go there.

(27) You should have gone to the store, but you didn't.

For this reason, we will assume a state-based R for epistemic modals but not for deontic ones:[23]

1. Epistemic modal verbs: R is state-based
2. Deontic modal verbs: R is possibly indisputable

As we will see, an indisputable R yields wide scope FC inferences for pragmatically enriched formulas:

- Wide scope FC: $[\Diamond\alpha \vee \Diamond\beta]^+ \models \Diamond\alpha \wedge \Diamond\beta$ [if R is indisputable]

These facts together give us the following predictions: wide scope FC is always predicted for epistemic modals, which, leading to epistemic contradiction, require an accessibility relation which is state-based and therefore indisputable. Deontic modals instead only lead to wide scope FC inference in certain contexts, namely when the assumption of indisputability is justified. These are contexts where the speaker is assumed to be fully informed about what is obligatory or allowed, for example in some performative uses of the verb. We will return to these predictions in section 6.1.

4.2 Pragmatic enrichment

The pragmatic enrichment function is recursively defined for formulas in the NE-free fragment of the language as follows:

Definition 6 (Pragmatic enrichment function)

$$\begin{aligned}
[p]^+ &= p \wedge \textsc{ne} \\
[\neg\alpha]^+ &= \neg[\alpha]^+ \wedge \textsc{ne} \\
[\alpha \vee \beta]^+ &= ([\alpha]^+ \vee [\beta]^+) \wedge \textsc{ne} \\
[\alpha \wedge \beta]^+ &= ([\alpha]^+ \wedge [\beta]^+) \wedge \textsc{ne} \\
[\Diamond\alpha]^+ &= \Diamond[\alpha]^+ \wedge \textsc{ne}
\end{aligned}$$

It is easy to see that pragmatic enrichment has a non-trivial effect on disjunctions. This effect in combination with our notion of modality allows us to derive FC inferences for pragmatically enriched formulas:

- Narrow scope FC: $[\Diamond(\alpha \vee \beta)]^+ \models \Diamond\alpha \wedge \Diamond\beta$
- Wide scope FC: $[\Diamond\alpha \vee \Diamond\beta]^+ \models \Diamond\alpha \wedge \Diamond\beta$ [if R is indisputable]

[23] An anonymous reviewer suggested that deontic modals might require that the accessible worlds are a subset of the information state. This would explain how deontic modals are sensitive to information as for example in miners style puzzles (Kolodny and MacFarlane, 2010) or other puzzles arising for deontic 'must' (Ninan, 2005). On the other hand, it is ok to say 'you are here but you may go there' and this would be unexplained if only epistemic possibilities are deontically accessible. I leave this issue to future investigation.

A second crucial result is that pragmatic enrichment has non-trivial effects *only* on disjunctions, and only if they occur in a positive environment. In particular pragmatic enrichment is vacuous under negation, which allows us to derive Dual Prohibition:

- Dual Prohibition: $[\neg\Diamond(\alpha \vee \beta)]^+ \models \neg\Diamond\alpha \wedge \neg\Diamond\beta$

Note however that in the scope of a double negation, FC effects arise again:

- Double Negation: $[\neg\neg\Diamond(\alpha \vee \beta)]^+ \models \Diamond\alpha \wedge \Diamond\beta$

Evidence for the correctness of the latter prediction comes from cases of presupposition projection discussed by Romoli and Santorio (2019). Romoli and Santorio consider examples like (28) (where X_P means that X triggers the presupposition P):

(28) a. Either Maria can't go to study in Tokyo or Boston, or she is the first in our family who can go to study in Japan (and the second who can go to study in the States).
b. $\neg\Diamond(a \vee b) \vee X_{\Diamond a}$

In (28), the second disjunct *She is the first in our family who can go to study in Japan* presupposes *She can go to study in Japan*, but this presupposition does not project, it is filtered by the negation of the first disjunct. Assuming that a disjunction $A \vee B_P$ presupposes $\neg A \to P$, the predicted presupposition for (28) is (29):

(29) $\neg\neg\Diamond(a \vee b) \to \Diamond a$

In a system deriving FC inferences and validating Double Negation elimination (such as BSML but also Willer (2018); Goldstein (2019) and others) (29) is trivially satisfied (double negations cancel each other out and free choice inference is computed), so the correct filtering is predicted.

All the mentioned results will be proven in the following section.

5 Results

5.1 Pragmatic enrichment, disjunction and modals

As mentioned in the previous sections, one of the main results of the present research is that pragmatic enrichment has a non-trivial effect in interaction with disjunction and only in interaction with disjunction. More precisely, we can prove the following facts. First, by an easy double induction we can prove Fact 1, which will be useful later on.

Fact 1

$$M, s \models [\alpha]^+ \;\Rightarrow\; M, s \models \alpha$$
$$M, s \dashv [\alpha]^+ \;\Rightarrow\; M, s \dashv \alpha$$

Proof: By double induction on the complexity of α.

1. $\alpha = p$
   - $M, s \models [p]^+ \Rightarrow M, s \models p \wedge \text{NE} \Rightarrow M, s \models p$
   - $M, s \dashv [p]^+ \Rightarrow M, s \dashv p \wedge \text{NE} \Rightarrow$ there are $t, t' : s = t \cup t'$ and $M, t \dashv p$ & $M, t' \dashv \text{NE}$ $\Rightarrow t' = \emptyset$ and $M, s \dashv p$
2. $\alpha = \neg\beta$
   - $M, s \models [\neg\beta]^+ \Rightarrow M, s \models \neg[\beta]^+ \wedge \text{NE} \Rightarrow M, s \dashv [\beta]^+ \Rightarrow_{IH} M, s \dashv \beta \Rightarrow M, s \models \neg\beta$
   - $M, s \dashv [\neg\beta]^+ \Rightarrow M, s \dashv \neg[\beta]^+ \wedge \text{NE} \Rightarrow M, s \dashv \neg[\beta]^+ \Rightarrow M, s \models [\beta]^+ \Rightarrow_{IH} M, s \models \beta$ $\Rightarrow M, s \dashv \neg\beta$
3. $\alpha = \beta \vee \gamma$

- $M, s \models [\beta \vee \gamma]^+ \Rightarrow M, s \models [\beta]^+ \vee [\gamma]^+ \Rightarrow$ there are $t, t' : s = t \cup t'$ and $M, t \models [\beta]^+$ & $M, t' \models [\gamma]^+ \Rightarrow_{IH} M, t \models \beta$ and $M, t' \models \gamma \Rightarrow M, s \models \beta \vee \gamma$
- $M, s \Dashv [\beta \vee \gamma]^+ \Rightarrow M, s \Dashv [\beta]^+$ and $M, s \Dashv [\gamma]^+ \Rightarrow_{IH} M, s \Dashv \beta$ and $M, s \Dashv \gamma \Rightarrow M, s \Dashv \beta \vee \gamma$

4. $\alpha = \beta \wedge \gamma$
   - $M, s \models [\beta \wedge \gamma]^+ \Rightarrow M, s \models [\beta]^+ \wedge [\gamma]^+$ $M, s \models [\beta]^+$ and $M, s \models [\gamma]^+ \Rightarrow_{IH} M, s \models \beta$ and $M, s \models \gamma \Rightarrow M, s \models \beta \wedge \gamma$
   - $M, s \Dashv [\beta \wedge \gamma]^+ \Rightarrow M, s \Dashv [\beta]^+ \wedge [\gamma]^+ \Rightarrow$ there are $t, t' : s = t \cup t'$ and $M, t \Dashv [\beta]^+$ & $M, t' \Dashv [\gamma]^+ \Rightarrow_{IH} M, t \Dashv \beta$ and $M, t' \Dashv \gamma \Rightarrow M, s \Dashv \beta \wedge \gamma$

5. $\alpha = \Diamond\beta$
   - $M, s \models [\Diamond\beta]^+ \Rightarrow M, s \models \Diamond[\beta]^+ \Rightarrow$ for all $w \in s$, there is a non-empty $t \subseteq R[w]$: $M, t \models [\beta]^+ \Rightarrow_{IH} M, t \models \beta \Rightarrow M, s \models \Diamond\beta$
   - $M, s \Dashv [\Diamond\beta]^+ \Rightarrow M, s \Dashv \Diamond[\beta]^+ \Rightarrow$ for all $w \in s$, $M, R[w] \Dashv [\beta]^+ \Rightarrow_{IH} M, R[w] \Dashv \beta \Rightarrow M, s \Dashv \Diamond\beta$

□

Then notice that from Definition 6 of Pragmatic Enrichment it directly follows that $[\alpha]^+ \models$ NE, and from Fact 1 it follows that $[\alpha]^+ \models \alpha$. So we have:

Fact 2 *Let α be* NE*-free.*

$$[\alpha]^+ \models \alpha \wedge \text{NE}$$

Another easy induction shows that if α does not contain any disjunction, then also the other direction holds:

Fact 3 *Let α be* NE*-free and $\vee$-free. Then*

$$\alpha \wedge \text{NE} \models [\alpha]^+$$

This means that if α is disjunction-free, the only difference between α and $[\alpha]^+$ is that the former is supported by the empty set, $\emptyset$, while the latter is not. The effect of pragmatic enrichment in these cases is trivial in the sense that it does not lead to any linguistically interesting prediction. It is only in interaction with disjunction that pragmatic enrichment leads to non-trivial results (and, as we will prove in section 5.2, only when disjunction occurs in a positive context). Let us have a closer look.

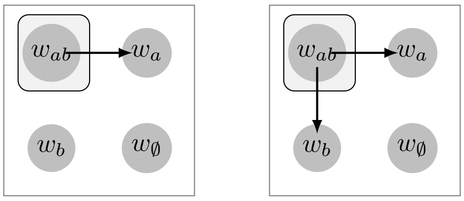
As we saw, in BSML a state s supports a plain disjunction iff s is the union of two substates, each supporting one of the disjuncts. The effect of pragmatically enriching $(\alpha \vee \beta)$ is that we add a conjunction with NE to each subformula of the original sentence. More precisely, $[\alpha \vee \beta]^+ = ([\alpha]^+ \vee [\beta]^+) \wedge \text{NE}$, which implies:

$$[\alpha \vee \beta]^+ \models (\alpha \wedge \text{NE}) \vee (\beta \wedge \text{NE})$$

Given the meaning of NE (only supported in non-empty states) the effect we obtain is that s supports $[\alpha \vee \beta]^+$ iff s is the union of two **non-empty** substates, each supporting one of the disjuncts. A pragmatically enriched disjunction $[\alpha \vee \beta]^+$ then requires both disjuncts to be live possibilities. See Figure 2 for illustrations. The notion of being a live possibility in a state can be expressed in BSML using the possibility modal $\Diamond$, if we assume a state-based accessibility relation.

Fact 4 (Modal disjunction) *Let S be the set of state-based model-state pairs.*

$$[\alpha \vee \beta]^+ \models_S \Diamond\alpha \wedge \Diamond\beta$$



(a) $\models \Diamond(a \vee b)$; but $\not\models \Diamond(a \vee b)^+$ (b) $\models \Diamond(a \vee b)^+$

Figure 4: Narrow scope FC derived for pragmatically enriched disjunction

Proof: Suppose $M, s \models [\alpha \vee \beta]^+$, which implies $M, s \models (\alpha \wedge \text{NE}) \vee (\beta \wedge \text{NE})$. The latter implies that there is a non-empty subset of s which supports α. By state-basedness, $R[w] = s$ for all $w \in s$, but then we have $M, s \models \Diamond\alpha$. By the same reasoning we conclude $M, s \models \Diamond\beta$, and therefore $M, s \models \Diamond\alpha \wedge \Diamond\beta$. □

When embedded under a modal, a pragmatically enriched disjunction gives rise to FC effects.

Fact 5 (Narrow scope FC)

$$[\Diamond(\alpha \vee \beta)]^+ \models \Diamond\alpha \wedge \Diamond\beta$$

Proof: Suppose $M, s \models [\Diamond(\alpha \vee \beta)]^+$, which means $M, s \models \Diamond(([\alpha]^+ \vee [\beta]^+) \wedge \text{NE}) \wedge \text{NE}$, which implies $M, s \models \Diamond([\alpha]^+ \vee [\beta]^+)$ and $s \neq \emptyset$. Let $w \in s$. $M, s \models \Diamond([\alpha]^+ \vee [\beta]^+)$ means that there is a non-empty $t \subseteq R[w]$ such that $M, t \models [\alpha]^+ \vee [\beta]^+$. Therefore there are some t_1,t_2 such that $t = t_1 \cup t_2$ and $M, t_1 \models [\alpha]^+$ and $M, t_2 \models [\beta]^+$. Since, by Fact 2, $[\alpha]^+ \models \text{NE}$ and $[\alpha]^+ \models \alpha$, it follows that $t_1 \neq \emptyset$ and $M, t_1 \models \alpha$. Since w was arbitrary we conclude $M, s \models \Diamond\alpha$. By the same reasoning we conclude $M, s \models \Diamond\beta$ and therefore $M, s \models \Diamond\alpha \wedge \Diamond\beta$. □

Consider Figure 4 for an illustration. The pragmatically enriched $[\Diamond(a \vee b)]^+$ is not supported in the state depicted in 4(a) because $R[w_{ab}]$ does not support $[a \vee b]^+$ since b is not an open possibility there. The classical $(a \vee b)$, instead, is supported in $R[w_{ab}]$ with $\emptyset$ as relevant substate supporting b, and therefore the state supports $\Diamond(a \vee b)$. In 4(b), instead, both a and b are open possibilities in $R[w_{ab}]$ and therefore $M, R[w_{ab}] \models [a \vee b]^+$ and so $M, s \models [\Diamond(a \vee b)]^+$.

From Fact 5 it directly follows that FC effects are also generated for disjunctions with logically dependent disjuncts and it is easy to see that the same holds for those with more than two disjuncts:

- $[\Diamond(\alpha \vee (\alpha \wedge \beta))]^+ \models \Diamond\alpha \wedge \Diamond(\alpha \wedge \beta)$
- $[\Diamond(\alpha \vee (\beta \vee \gamma))]^+ \models \Diamond\alpha \wedge \Diamond\beta \wedge \Diamond\gamma$

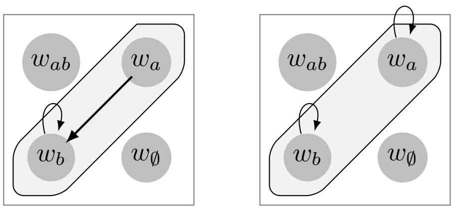
Wide scope FC instead is only derived if we assume an indisputable accessibility relation.

Fact 6 (Wide scope FC) *Let I be the set of indisputable model-state pairs.*

$$[\Diamond\alpha \vee \Diamond\beta]^+ \models_I \Diamond\alpha \wedge \Diamond\beta$$

Proof: Suppose $M, s \models [\Diamond\alpha \vee \Diamond\beta]^+$, which implies $M, s \models [\Diamond\alpha]^+ \vee [\Diamond\beta]^+$ and $s \neq \emptyset$. It follows that $s = t' \cup t''$ for some non-empty t' and t'' such that $M, t' \models \Diamond\alpha$ and $M, t'' \models \Diamond\beta$. Since R is indisputable, $R[w] = R[v]$ for all $w, v \in s$. But then $M, t' \models \Diamond\alpha$ implies $M, s \models \Diamond\alpha$ and $M, t'' \models \Diamond\beta$ implies $M, s \models \Diamond\beta$, and so $M, s \models \Diamond\alpha \wedge \Diamond\beta$. □

Pragmatic enrichment and indisputability are both needed to obtain wide scope FC inference, as illustrated in Figure 5. The accessibility relation in 5(a) is indisputable but wide scope FC



(a) $\models \Diamond a \vee \Diamond b$, but $\not\models \Diamond a$

(b) $\models (\Diamond a \vee \Diamond b)^+$, but $\not\models \Diamond a$

Figure 5: Failures of wide scope FC.

fails since the disjunction is not pragmatically enriched and so the substate of s supporting $\Diamond a$ can be empty. In 5(b), instead, wide scope FC fails because R is not indisputable, and so while we have non-empty subsets of s supporting each disjunct, as required by the pragmatically enriched disjunction, s itself need not support them.

5.2 Bilateralism, negation and contradictions

In bilateral systems negation is taken to derive its meaning from the speech act of rejection, assumed to be a primitive notion (Smiley, 1996; Rumfitt, 2000; Incurvati and Schlöder, 2017). In BSML rejection is represented by $\dashv$:

$$\begin{array}{lcl} M, s \models \neg\phi & \text{iff} & M, s \dashv \phi \\ M, s \dashv \neg\phi & \text{iff} & M, s \models \phi \end{array}$$

One might argue that allowing to pre-encode what should happen under negation, bilateral systems like ours are more descriptive than explanatory (see Aloni, 2018; Bar-Lev and Fox, 2020). However our choices with respect to the anti-support clauses have not been arbitrary: if negation is grounded on the more primitive notion of rejection, it is speakers' behaviour which determines what happens under negation. Furthermore bilateral systems are not merely descriptive: their explanatory power comes with their overall predictions, which transcend the choices made with respect to the basic cases. In this subsection I will present some of these predictions, in particular those arising from the interaction between $\neg$ and our operation of pragmatic enrichment. But first I will discuss a number of general properties of negation in BSML.

First of all, it is easy to see that the current semantics validates a number of classical laws typical of Boolean negation ($\phi \equiv \psi$ is short for $\phi \models \psi$ and $\psi \models \phi$):

Fact 7 (Classical validities)

$$\begin{array}{rcll} \phi & \equiv & \neg\neg\phi & \textit{(Double Negation Elimination)} \\ \neg(\phi \vee \psi) & \equiv & \neg\phi \wedge \neg\psi & \textit{(De Morgan Laws)} \\ \neg(\phi \wedge \psi) & \equiv & \neg\phi \vee \neg\psi & \\ \neg\Box\phi & \equiv & \Diamond\neg\phi & \textit{(Duality } \Box/\Diamond\textit{)} \\ \neg\Diamond\phi & \equiv & \Box\neg\phi & \end{array}$$

Furthermore, our notion of negation is closely connected with the notion of incompatibility, a connection which has been identified as the hallmark of negation (Berto, 2015). In BSML we can show that if a state s supports a negative sentence $\neg\phi$, then s is incompatible with any state supporting ϕ, where being incompatible means having an empty intersection (for a proof see Anttila 2021, Proposition 3.3.9, page 53.)

Fact 8 (Negation and incompatibility)

$$M, s \models \neg\phi \;\Rightarrow s \cap t = \emptyset, \textit{ for all } t \textit{ such that } M, t \models \phi$$

Incompatibility however does not define negation in BSML. The converse of Fact 8 does not hold. As a counterexample consider $\phi := \neg((p \wedge \text{NE}) \vee q)$. Let $\{w_q\}$ be a state consisting of a single world where q is true and p is false. Then $\{w_q\}$ is incompatible with any state supporting ϕ: $\{w_q\} \cap t = \emptyset$, for all t such that $M, t \models \phi$, but $M, \{w_q\} \not\models \neg\phi$, because $\neg\phi$ is equivalent to $(p \wedge \text{NE}) \vee q$, and p is not an open possibility in $\{w_q\}$.

Our counterexample involves the non-emptiness atom NE.[24] In interaction with NE, bilateral negation gives rise to further non-classical behaviour, including a failure of replacement,[25] and, therefore, a failure of the law of contraposition. As we will see, however, this non-classical behaviour is precisely what we need to explain the effects of pragmatic enrichment in negative contexts.[26]

I will first show how bilateral negation leads to a failure of replacement in interaction with NE and then discuss how precisely this fact leads to correct predictions when bilateral negation applies to pragmatically enriched formulas.

5.2.1 Tautologies and contradictions

In BSML we can distinguish between strong and weak notions of tautologies and contradictions. While the weak tautology, NE, is supported by every non-empty states, the strong tautology (the classical $p \vee \neg p$) is supported by *every* state, including the empty state. And while a weak contradiction (both classical $p \wedge \neg p$ and $\neg$NE) is supported *only* by the empty state (the former because $\emptyset$ supports all classical formulas including p and $\neg p$), a strong contradiction, requiring an empty and non-empty supporting state is never supported.

- Weak
  - $\top := \text{NE}$ — supported by all non-empty states
  - $\bot_1 := p \wedge \neg p$ — supported only by the empty state
  - $\bot_2 := \neg\text{NE}$ — supported only by the empty state
- Strong
  - $\mathbf{1} := p \vee \neg p$ — always supported
  - $\mathbf{0_1} := \text{NE} \wedge \bot_1$ — never supported
  - $\mathbf{0_2} := \text{NE} \wedge \bot_2$ — never supported

Weak contradictions $\bot_1$ and $\bot_2$ are mutually equivalent (supported only by the empty state) but behave differently under negation. Similarly for the strong contradictions $\mathbf{0_1}$ and $\mathbf{0_2}$.

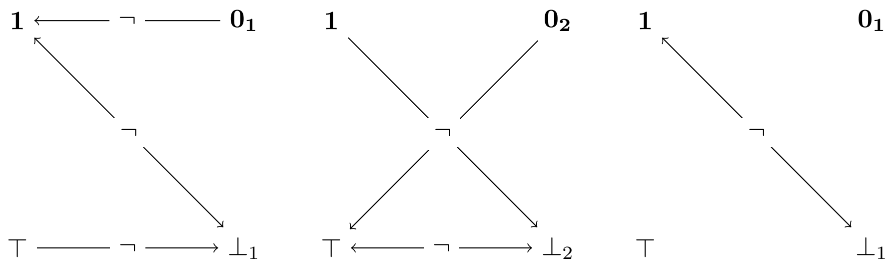
- $\bot_1 \equiv \bot_2$, but $\neg\bot_1 \not\equiv \neg\bot_2$;
- $\mathbf{0_1} \equiv \mathbf{0_2}$, but $\neg\mathbf{0_1} \not\equiv \neg\mathbf{0_2}$.

These facts are examples of failures of replacement under negation.

[24] Other courterexamples, not involving NE, can be constructed in a language including the dependence atoms or inquisitive disjunction (Kontinen and Väänänen, 2011).

[25] This means that to know the set of states which satisfy ϕ is not enough to know the set of states which satisfy $\neg\phi$. Therefore, bilateral negation is not a semantic operation in the sense of (Burgess, 2003; Kontinen and Väänänen, 2011).

[26] As correctly observed by an anonymous reviewer when replacement under $\neg$ holds, the rejection conditions are derivable from the support conditions, and so without failure of replacement bilateralism would not give different predictions from unilateral systems and so it would be empirically unjustified. Note however that there have been conceptual arguments in favor of bilateralism purely based on classical logic where replacement holds (e.g., Smiley, Rumfitt and others) because bilateralism for example allows for a harmonious proof theory.



(a) Negation Square 1 (b) Negation Square 2 (c) Negation Square 3

Figure 6: Negation Squares

Fact 9 (Failure of Replacement under $\neg$)

$$\phi \equiv \psi \not\Rightarrow \neg\phi \equiv \neg\psi$$

The different behaviour of weak and strong contradictions under negation is summarised in Figure 6. $\neg\bot_1$ is different from $\neg\bot_2$ because the former is equivalent to $\mathbf{1}$, a strong tautology, while the latter is equivalent to $\top$, a weak tautology, and $\mathbf{1} \not\equiv \top$ (the former is supported by any state, the latter only by all *non-empty* states). Similarly for $\mathbf{0_1}$ and $\mathbf{0_2}$:

- $\neg\bot_1 = \neg(p \wedge \neg p) \equiv p \vee \neg p = \mathbf{1} \not\equiv \top = \textsc{ne} \equiv \neg\neg\textsc{ne} = \neg\bot_2$
- $\neg\mathbf{0_1} \equiv \neg(\textsc{ne} \wedge (p \wedge \neg p)) \equiv p \vee \neg p \equiv \mathbf{1} \not\equiv \top = \textsc{ne} \equiv \neg(\textsc{ne} \wedge \neg\textsc{ne}) = \neg\mathbf{0_2}$

Furthermore, although strong and weak tautologies are different, their negations are equivalent:

- $\neg\mathbf{1} = \neg(p \vee \neg p) \equiv p \wedge \neg p \equiv \neg\textsc{ne} = \neg\top$

In combination with the soundness of double negation elimination this leads (again) to a failure of replacement under $\neg$:

- $\neg\mathbf{1} \equiv \neg\top$, but $\neg\neg\mathbf{1} \not\equiv \neg\neg\top$

The cause of all problems seems to be NE when occurring under negation. For this reason in dependence/team logic negation is usually only defined for the classical fragment of the language. This would give us the "well-behaving" Negation Square 3 in Figure 6(c). But this strategy is not an option for us. The effect of pragmatic enrichment (formulated in terms of NE) in negative sentences is among the phenomena that motivated this research. In what follows I will show that despite (or better because of) the logical misbehaviour of NE under $\neg$, our predictions with respect to the pragmatic enrichment of negative formulas are in agreement with our natural language intuitions.

5.2.2 Pragmatic enrichment under negation

First of all, we have already seen that FC effects disappear under negation (Dual Prohibition facts), so pragmatic enrichment should be vacuous in such an environment:

$$\neg[\alpha]^+ \equiv \neg\alpha$$

On the other hand, as Romoli and Santorio (2019) argued, there is evidence showing that speakers draw FC inferences under double negation (Double Negation facts), so pragmatic enrichment should not be vacuous there:

$$\neg\neg[\alpha]^+ \not\equiv \neg\neg\alpha$$

Without replacement failure these two desiderata would be impossible to satisfy.

Before proving these facts a clarification: the pragmatic enrichment of a negative sentence is defined as follows:

$$[\neg\alpha]^+ = \neg[\alpha]^+ \wedge \text{ NE}$$

When we say that pragmatic enrichment is vacuous under negation we do not mean to say that pragmatically enriching a negative sentence is always trivial. For example, a pragmatically enriched negative disjunct can give rise to free choice effects:

$$[\neg\alpha]^+ \vee \psi \not\equiv (\neg\alpha \vee \psi)$$

What we mean instead is that if α is a positive sentence occurring under a negation, pragmatically enriching α is vacuous there.

Fact 10 (Pragmatic enrichment vacuous under single negation) *Let α be a positive sentence, then*

$$\neg[\alpha]^+ \equiv \neg\alpha$$

Proof: By induction on the complexity of α. We only give two instructive cases:

- $\alpha = p$. $\neg[p]^+ \equiv \neg(p \wedge \text{NE}) \equiv \neg p \vee \neg\text{NE} \equiv \neg p$
- $\alpha = \beta \vee \gamma$. $\neg[\beta \vee \gamma]^+ \equiv \neg(([\beta]^+ \vee [\gamma]^+) \wedge \text{NE}) \equiv \neg([\beta]^+ \vee [\gamma]^+) \vee \neg\text{NE} \equiv \neg([\beta]^+ \vee [\gamma]^+) \equiv \neg[\beta]^+ \wedge \neg[\gamma]^+ \equiv_{\text{IH}} \neg\beta \wedge \neg\gamma \equiv \neg(\beta \vee \gamma)$

The crucial fact exploited in the proof of Fact 10 is that $\neg(\phi \wedge \text{NE})$ is by the de Morgan laws equivalent to $\neg\phi \vee \neg\text{NE}$, which, since only the empty set supports $\neg\text{NE}$, is equivalent to $\neg\phi$.

- $\neg(\phi \wedge \text{NE}) \equiv \neg\phi \vee \neg\text{NE} \equiv \neg\phi$

Note however that if $\neg(\phi \wedge \text{NE})$ occurs under another negation, things change. From $\neg\neg(\phi \wedge \text{NE})$, by double negation elimination, we obtain $(\phi \wedge \text{NE})$ which is not equivalent with ϕ or $\neg\neg\phi$.

- $\neg\neg(\phi \wedge \text{NE}) \equiv \phi \wedge \text{NE} \not\equiv \phi \equiv \neg\neg\phi$

Pragmatic enrichment then is not vacuous under double negation.

Fact 11 (Pragmatic enrichment not vacuous under double negation)

$$\neg[\neg\alpha]^+ \equiv \neg\neg[\alpha]^+ \not\equiv \neg\neg\alpha$$

Proof: Here is a counterexample:

$$\neg[\neg p]^+ \equiv \neg(\neg[p]^+ \wedge \text{ NE}) \equiv \neg\neg[p]^+ \vee \neg\text{NE} \equiv \neg\neg[p]^+ \equiv \neg\neg(p \wedge \text{NE}) \equiv p \wedge \text{NE} \not\equiv p \equiv \neg\neg p$$

So we have another example of a failure of replacement under negation:

- $\neg[p]^+ \equiv \neg p$ (Fact 10), but $\neg\neg[p]^+ \not\equiv \neg\neg p$ (Fact 11).

As a corollary of Fact 10 we obtain Dual Prohibition for positive α and β. The proof for arbitrary α and β also uses Fact 1:

Fact 12 (Dual Prohibition)

$$[\neg\Diamond(\alpha \vee \beta)]^+ \models \neg\Diamond\alpha \wedge \neg\Diamond\beta$$

Proof: Suppose $M, s \models [\neg\Diamond(\alpha \vee \beta)]^+$, which implies $M, s \Dashv \Diamond[\alpha \vee \beta]^+$ and $s \neq \emptyset$. Let $w \in s$. Then $M, R[w] \Dashv [\alpha \vee \beta]^+$, which means $M, R[w] \Dashv ([\alpha]^+ \vee [\beta]^+) \wedge$ NE, which implies $M, R[w] \Dashv [\alpha]^+ \vee [\beta]^+$. This means that $M, R[w] \Dashv [\alpha]^+$ and $M, R[w] \Dashv [\beta]^+$, which by Fact 1 implies $M, R[w] \Dashv \alpha$ and $M, R[w] \Dashv \beta$. But then since w was arbitrary, $M, s \Dashv \Diamond\alpha$ which implies $M, s \models \neg\Diamond\alpha$ and $M, s \Dashv \Diamond\beta$, which implies $M, s \models \neg\Diamond\beta$, and so $M, s \models \neg\Diamond\alpha \wedge \neg\Diamond\beta$. □

Double Negation follows from Fact 5 and Double Negation Elimination:

Fact 13 (Double Negation)

$$[\neg\neg\Diamond(\alpha \vee \beta)]^+ \models \Diamond\alpha \wedge \Diamond\beta$$

6 Applications

6.1 Epistemic and Deontic Free Choice

As we saw, BSML derives both narrow scope and wide scope FC effects for pragmatically enriched sentences, but while narrow scope effects are generated for any modality, wide scope FC arises only in case the modality is of the indisputable kind.

1. $[\Diamond(\alpha \vee \beta)]^+ \models \Diamond\alpha \wedge \Diamond\beta$
2. $[\Diamond\alpha \vee \Diamond\beta]^+ \models \Diamond\alpha \wedge \Diamond\beta$ [if R is indisputable]

Since state-based accessibility relations are also indisputable, narrow and wide scope FC are always predicted for epistemic modal verbs, which involve a state-based R:

(30) He might either be in London or in Paris. [+fc, narrow]

(31) He might be in London or he might be in Paris. [+fc, wide]

The case of deontic FC is more subtle. Assuming that deontics trigger an indisputable R only in certain contexts, namely when the speaker is assumed to be knowledgeable about what is permitted/obligatory (e.g. in performative uses),[27] we make the following predictions:

- narrow scope FC is always predicted for deontics
- wide scope FC is predicted only if speaker knows what is permitted/obligatory

These predictions have received preliminary confirmations from experiments reported in Cremers *et al.* (2017). In these experiments, judgements on free choice effects were collected, for both wide and narrow scope disjunctions, in different contexts with the speaker assumed to be knowledgeable or not. To distinguish narrow scope from wide scope configurations, examples like the following were used, where the position of *either* arguably constrains the syntactic scope of *or*:

(32) We may either eat the cake or the ice cream. [narrow scope disjunction favoured, but not forced]

(33) Either we may eat the cake or the ice cream. [wide scope disjunction forced]

More specifically, as argued by Larson (1985), the high position of *either* in (33) forces a wide scope disjunction configuration, while its low position in (32) favours a narrow scope interpretation. One rather surprising result of these experiments was that only in wide scope configurations like (33) the availability of the FC inference was dependent on the assumption on speaker knowledge, exactly as predicted by the present analysis.

[27] An anonymous reviewer correctly noticed that authority or knowledgableness does not always guarantee indisputability. Here is their example: "Assume I am an authority on traffic laws, but I am epistemically uncertain about whether the one way street at the end of the block runs east or west. Then my deontic accessibility relation will map some of the worlds in my information state onto possibilities in which you may turn left, and other worlds onto possibilities in which you may turn right. So I am fully knowledgable with respect to permissible actions, but my deontic accessibility relation is not indisputable." In the scenario described, the relevant agent is knowledgeable in the sense that she knows the general traffic rules, but she is not knowledgable in the stronger sense of knowing what propositions are obligatory/allowed, e.g., she does not know whether one is allowed to turn left or right. It is this stronger sense of knowledgableness, captured by indisputability, which is relevant for wide scope free choice. Indeed, in the scenario above, "You may turn left or you may turn right" does not lead to the permission to do either.

A further consequence of our analysis is that all cases of overt free choice cancellation must be treated as examples of wide scope disjunction. This is arguably the case for examples like (34) involving sluicing, as discussed in Fusco (2019):

(34) You may eat the cake or the ice-cream, I don't know which.

Whether the same assumption is also justified in cases like (35), discussed by Kaufmann (2016), must be left to another occasion:

(35) You may either eat the cake or the ice-cream, it depends on what John has taken.

If we assumed that sluicing always requires wide scope disjunctions as antecedents, the first sentence in (36) would also be a case of a wide scope disjunction:

(36) You may either eat the cake or the ice-cream, I don't care which.

But then we would predict that FC inferences would be generated only in contexts where the speaker is assumed to be knowledgeable and this prediction does not seem to be correct. (36) seems to trigger a FC inference no matter what. Notice however that as Fusco argues there is a difference in the elided material of (34) and (36), as illustrated by the following pair:

(37) You may either eat the cake or the ice-cream, I don't know which you may eat.

(38) You may either eat the cake or the ice-cream, I don't care which you eat.

But then we can assume that the sluicing construction in (36) requires a disjunction of the form 'You eat the cake or you eat the ice-cream' as antecedent (rather than 'You may eat the cake or you may eat the ice-cream') and so triggers a narrow scope disjunction configuration in the first sentence, which in our system always gives rise to free choice effects.

6.2 Ignorance and Obviation

While all human languages appear to contain a word for negation, there are various examples of languages lacking explicit coordination structures. In these languages there is no word corresponding to *or*, but disjunctive meanings can typically still be expressed for example by adding a suffix/particle expressing uncertainty to the main verb. Example (39) illustrates this strategy for Maricopa (a Yuman language of Arizona described by Gil (1991)):

(39) Johnš Billš v?aawuumšaa.
John-nom Bill-nom 3-come-pl-fut-infer
'John or Bill will come'

(40) Johnš Billš v?aawuum.
John-nom Bill-nom 3-come-pl-fut
'John and Bill will come' [Maricopa, Gil 1991, p. 102]

In (39) the "uncertainty" suffix *šaa* is added to the main verb and it is what triggers a disjunctive interpretation. Indeed when omitted as in (40) the interpretation of the sentence becomes conjunctive. This example provides evidence of the close connection between disjunction and uncertainty. Plain disjunctions give rise to ignorance effects (Grice, 1989; Gazdar, 1979):

(41) Ignorance
a. John has two or three children.
$\rightsquigarrow$ speaker doesn't know how many
b. $\alpha \vee \beta \rightsquigarrow \Diamond\alpha \wedge \Diamond\beta$ [epistemic $\Diamond$]

These effect are quite strong as evidenced by the oddity of sentences like (42):

(42) ?I have two or three children.

(42) is odd because it suggests that the speaker doesn't know how many children she has, an implausible assumption. The ignorance effect (and therefore the oddity) disappear when we embed the disjunction under a universal quantifier:

(43) Obviation

a. Every woman in my family has two or three children.
$\not\rightsquigarrow$ speaker doesn't know how many children each woman has

b. $\forall x(\alpha \vee \beta) \not\rightsquigarrow \forall x(\Diamond\alpha \wedge \Diamond\beta)$ [epistemic $\Diamond$]

Disjunctions under universal quantifiers have been argued instead to give rise to a distribution inference (Fox, 2007; Klinedinst, 2006):

(44) Distribution

a. Every woman in my family has two or three children.
$\rightsquigarrow$ some woman has two and some woman has three

b. $\forall x(\alpha \vee \beta) \rightsquigarrow \exists x\alpha \wedge \exists x\beta$

The challenge here is to account for the ignorance inference in (42) and its obviation in (43), as well the distribution effect in (44). As we saw, BSML derives modal effects for pragmatically enriched plain disjunctions, which in combination with an exclusivity implicature, derives the desired ignorance inference:

- $[\alpha \vee \beta]^+ \models \Diamond\alpha \wedge \Diamond\beta$ [if R is state-based]

Aloni and van Ormondt (2021) discuss a first order extension of BSML, where states are defined as sets of world-assignment pairs, which also accounts for the obviation and distribution cases in (43) and (44), but the latter only in situations of maximal information (i.e. where s is a singleton set). In situations of partial information only the weaker inference in (45) is derived:

(45) Distribution (partial information)

a. Every woman in my family has two or three children.
$\rightsquigarrow$ some woman might have two and some woman might have three

b. $\forall x(\alpha \vee \beta) \rightsquigarrow \exists x\Diamond\alpha \wedge \exists x\Diamond\beta$

Finally notice that in this first order extension also the so-called ALL-OTHERS-FREE-CHOICE and ALL-OTHERS-DUAL-PROHIBITION readings for the following sentences, recently discussed by Gotzner *et al.* (2020), can be derived (as in Goldstein, 2019):

(46) Exactly one girl can take Spanish or Calculus. $\rightsquigarrow$ One girl can choose between the two and each of the others can take neither of them ALL-OTHERS-DUAL-PROHIBITION

(47) Exactly one girl cannot take Spanish or Calculus. $\rightsquigarrow$ One girl can take neither of the two and each of the others can choose between them ALL-OTHERS-FREE-CHOICE

6.3 Negative Free Choice

In BSML logically equivalent sentences can have different pragmatic effects. Although $\neg\alpha \vee \neg\beta$ and $\neg(\alpha \wedge \beta)$ are logically equivalent only disjunctions give rise to FC or other modal effects, which means that on our account these inferences are *detachable*:

$$\begin{array}{rcl} [\neg\alpha \vee \neg\beta]^+ & \models & \Diamond\neg\alpha \\ [\neg(\alpha \wedge \beta)]^+ & \not\models & \Diamond\neg\alpha \end{array}$$

$$\begin{array}{rcl} [\Diamond(\neg\alpha \vee \neg\beta)]^+ & \models & \Diamond\neg\alpha \\ [\Diamond\neg(\alpha \wedge \beta)]^+ & \not\models & \Diamond\neg\alpha \end{array}$$

Our predictions here differ from those of the implicature account of free choice (e.g., Fox, 2007), but also from Willer (2018). Instead they match the predictions of other inquisitive accounts (Aher, 2012; Ciardelli *et al.*, 2018c), but in contrast to inquisitive accounts, BSML need not assume a failure of the de Morgan Laws to obtain these results.

$$\begin{array}{rcl} \neg\alpha \vee \neg\beta & \equiv & \neg(\alpha \wedge \beta) \\ [\neg\alpha \vee \neg\beta]^+ & \not\equiv & [\neg(\alpha \wedge \beta)]^+ \end{array}$$

$$\begin{array}{rcl} \Diamond(\neg\alpha \vee \neg\beta) & \equiv & \Diamond\neg(\alpha \wedge \beta) \\ [\Diamond(\neg\alpha \vee \neg\beta)]^+ & \not\equiv & [\Diamond\neg(\alpha \wedge \beta)]^+ \end{array}$$

Ciardelli *et al.* (2018c) argued for the correctness of their prediction based on examples like (48):

(48) a. Mary might not speak both Arabic and Bengali.
b. #So, she might not speak Arabic.

In the current literature, however the empirical status of these inferences is debated. Marty *et al.* (2021) experimentally investigated the inference in (49) and concluded that such inference exists but is far less robust than plain positive (narrow scope) FC:

(49) Negative FC
a. It is not required that Mia buys apples and bananas. $\rightsquigarrow$ It is not required that Mia buys apples and that Mia buys bananas
b. $\neg\Box(\alpha \wedge \beta) \rightsquigarrow \neg\Box\alpha \wedge \neg\Box\beta$

In BSML, given the duality of $\Box$ and $\Diamond$ we have that $[\Diamond\neg(\alpha \wedge \beta)]^+ \equiv [\neg\Box(\alpha \wedge \beta)]^+$ and thus Negative FC is not predicted:

- Negative FC: $[\neg\Box(\alpha \wedge \beta)]^+ \not\models \neg\Box\alpha$

To account for the different strength of Negative and Positive FC we could assume that the former is an indirect scalar implicature effect while the latter is a pragmatic enrichment effect, similarly to what Marty *et al.* (2021) call a hybrid approach to these facts. In what follows I will explore another possible explanation.

6.3.1 Syntactic vs model-theoretic characterisation of neglect-zero

So far we have considered a syntactic characterisation of neglect-zero effects, via the pragmatic enrichment function $[\ ]^+$ defined in terms of NE. But neglect-zero effects can also be derived model-theoretically by ruling out $\emptyset$ from the set of the possible states. In this section we will compare these two strategies. Let BSML$^+$ be the pragmatically enriched version of BSML, i.e. BSML with pragmatic enrichment applied globally:

$$\alpha \models_{BSML^+} \beta \text{ iff } [\alpha]^+ \models_{BSML} [\beta]^+$$

			BSML$^+$	BSML*
Positive FC	$\Diamond(\alpha \vee \beta) \rightsquigarrow \Diamond\alpha \wedge \Diamond\beta$	**strong**	+	+
Negative FC	$\neg\Box(\alpha \wedge \beta) \rightsquigarrow \neg\Box\alpha \wedge \neg\Box\beta$	**weak**	-	+
Modal Disjunction	$\alpha \vee \beta \rightsquigarrow \Diamond\alpha \wedge \Diamond\beta$	**weak**	+	+

Table 3: Comparison BSML$^+$, BSML* & experimental findings (Tieu *et al.*, 2019; Marty *et al.*, 2021).

Let BSML* be like BSML, with the only difference that in BSML* the empty set, $\emptyset$, is not among the possible states. Then the following fact holds:

Fact 14 *Let α, β be classical positive formulas (i.e., without* NE *or* $\neg$*).*

$$\alpha \models_{BSML^*} \beta \quad \textit{iff}\ \alpha \models_{BSML^+} \beta$$

If we restrict attention to positive formulas, the effect of pragmatic enrichment (disallowing the empty state syntactically via NE) is equivalent to that of ruling out the empty state model-theoretically. This equivalence however does not hold for arbitrary formulas. More specifically, in BSML*, FC inferences are generated also for negative conjunctions, which means that in BSML* we predict the validity of Negative FC:

Fact 15 (Negative FC)

$$\begin{array}{lll} \Diamond\neg(\alpha \wedge \beta) & \models_{BSML^*} & \Diamond\neg\alpha \\ \Diamond\neg(\alpha \wedge \beta) & \not\models_{BSML^+} & \Diamond\neg\alpha \\ & & \\ \neg\Box(\alpha \wedge \beta) & \models_{BSML^*} & \neg\Box\alpha \\ \neg\Box(\alpha \wedge \beta) & \not\models_{BSML^+} & \neg\Box\alpha \end{array}$$

It is tempting to try to account for the experimental findings of Marty *et al.* (2021) in terms of a contrast between a syntactic and a model-theoretical implementation of neglect-zero effects as modelled by BSML$^+$ and BSML* respectively. BSML* could represent the interpretation strategy adopted by the participants drawing positive and negative FC inferences (a minority), while BSML$^+$ could model the one of the participants drawing only positive FC inferences. There is however a striking parallelism between the experimental results on Negative FC from Marty *et al.* (2021) and those concerning ignorance inferences triggered by plain disjunction as reported by Tieu *et al.* (2019). In the control part of one of their experiments (see Tieu *et al.*, 2019, Figure 5), participants were asked to judge sentences like 'Angie bought the boat or the car', in a context where only one of the disjuncts was true, e.g., where Angie bought only the boat, so a situation where the sentence 'should have been judged as clearly true' (Tieu *et al.*, 2019, page 14). However, this context elicited a relatively large proportion of intermediate responses, which provides evidence for the reality of the modal disjunction inference. According to the results of these experiments, however, modal disjunction is considerably less available than positive FC, similarly to how free choice inferences are less available in their negative than in their positive form. An explanation of these experimental results plainly in terms of different BSML* and BSML$^+$ strategies, which would explain the different robustness of Positive and Negative FC, would however not explain the contrast between Positive FC and Modal Disjunction, as illustrated in Table 3. We will return to this issue in the next section. There we will conjecture that neglect-zero can give rise to two kinds of effects: (i) weak and *global* effects modelled by BSML*; and (ii) more robust effects possibly triggered by the lexical semantics of certain expressions modelled by *local* applications of the $[\ ]^+$-function as defined in BSML. In fact, while only BSML* predicts Negative FC, only our original BSML, with $\emptyset$ among its building blocks, can model such local neglect-zero effects as well as their global suspension.

		$BSML^{\emptyset}$	$BSML^{+}$
Positive FC	$\Diamond(\alpha \vee \beta) \rightsquigarrow \Diamond\alpha \wedge \Diamond\beta$	-	+
Addition	$\alpha \models \alpha \vee \beta$	+	-
Disjunctive syllogism	$(\alpha \vee \beta) \wedge \neg\alpha \models \beta$	+	-
Contraposition	$\alpha \models \beta \;\Rightarrow\; \neg\beta \models \neg\alpha$	+	-

Table 4: Comparison $BSML^{\emptyset}$ and $BSML^{+}$.

7 Suspension and locality of neglect-zero effects

One important issue arising from the present work concerns the why and when of the pragmatic effects modelled in (variants of) BSML. Why do they occur? Are they optional or obligatory? And if optional, can they be suspended locally or only globally?

Let me repeat my conjecture concerning the why question. The pragmatic enrichments formalised in BSML are clearly not the result of conversational reasonings possibly operating on embedded constituents (as for example in Simons, 2010), neither, as I will argue below, they result from spontaneous optional applications of some grammatical operators (as in the grammatical view championed by Chierchia *et al.*, 2011). Rather, I propose, they are a straightforward consequence of something else speakers do in conversation, namely, as explained above, when interpreting a sentence they create pictures of the world and in doing so they disfavor what I called zero-models, i.e. models which verify the relevant sentences by virtue of some empty configuration. Zero-models are disfavored because of their cognitive costs, confirmed by various experimental research.

Despite their cognitive cost, however, zero-models are not always neglected. In logico-mathematical reasonings, as in the following, neglect-zero effects are suspended:

(50) A. Therefore, A or B.

(51) A or B. Not A. Therefore, B.

(52) If A then B. Therefore, if not B then not A.

The reasonings in (50)-(52) crucially rely on the availability of zero-models. Therefore we certainly should allow for *global suspension* of neglect-zero enrichments in certain language games, for example in the context of a mathematical proof. In our framework, this logico-mathematical reasoning style is modelled by the NE-free fragment of BSML, which behaves like classical modal logic:

Fact 16 *Let $\models_{CML}$ denote logical consequence as defined in classical modal logic. Then for* NE*-free α, β:*

$$\alpha \models_{BSML} \beta \Leftrightarrow \alpha \models_{CML} \beta$$

I will call $BSML^{\emptyset}$, the NE-free fragment of BSML. Table 4 compares the prediction of $BSML^{\emptyset}$ and $BSML^{+}$. In $BSML^{\emptyset}$, the empty state and more in general zero-models are allowed and play an essential role. We can imagine that some reasoners "engage" with zero-models more often and easily than others, possibly depending on age or some other factors that would be interesting to explore in future experimental research. Maybe, paraphrasing Whitehead, the use of zero-models 'is only forced on us by the needs of cultivated modes of thought'.

> 'The point about zero is that we do not need to use it in the operations of daily life. No one goes out to buy zero fish. It is in a way the most civilized of all the cardinals, and its use is only forced on us by the needs of cultivated modes of thought.' (Alfred North Whitehead as quoted by Nieder 2016).

			BSML^{lex}	BSML^{*}
Positive FC	$\Diamond(\alpha \vee \beta) \rightsquigarrow \Diamond\alpha \wedge \Diamond\beta$	**strong**	+	+
Negative FC	$\neg\Box(\alpha \wedge \beta) \rightsquigarrow \neg\Box\alpha \wedge \neg\Box\beta$	**weak**	-	+
Modal Disjunction	$\alpha \vee \beta \rightsquigarrow \Diamond\alpha \wedge \Diamond\beta$	**weak**	-	+

Table 5: Comparison BSML^{lex} and BSML^{*}.

Global suspension of neglect-zero effects is then possible, as witnessed by mathematical reasoning, but do we also have local suspensions as for example would be predicted if $[\ ]^+$ were a grammatical operation, like EXH in localist accounts of scalar implicatures? At the end, this is an empirical question, which requires further investigation but let me make some preliminary remarks. Suppose $[\ ]^+$ were indeed a grammatical operation which can be optionally applied. On this view, $\Diamond([\alpha]^+ \vee \beta)$ and $\Diamond(\alpha \vee [\beta]^+)$ would be predicted as possible readings of 'You may do α or β'. This prediction does not seem to be correct. Mom's reaction in the following dialogue is incoherent:

(53) MOM: You may do your homework or go to the beach.
SON: Ok, then I go to the beach.
MOM: No I only meant that you may do your homework.

Example (53) suggests that $[\ ]^+$ is not an optional grammatical operation. The following example however is potentially problematic for a view which only allows global suspensions because it employs a mixture of logical reasoning and pragmatically enriched language use:

(54) a. I may do A or B, or I may do C. I may not do C. Therefore, I may do A and I may do B.
b. $([\Diamond(\alpha \vee \beta)]^+ \vee \Diamond\gamma) \wedge \neg\Diamond\gamma \models \Diamond\alpha \wedge \Diamond\beta$

If judged valid, the reasoning in (54) would provide evidence for an analysis which assumes that modal verbs (obligatorily) trigger pragmatic enrichment in their prejacents as part of their lexical meaning. (54) would be an example of the valid inference in (55):

(55) $(\Diamond[\alpha \vee \beta]^+ \vee \Diamond[\gamma]^+) \wedge \neg\Diamond[\gamma]^+ \models \Diamond[\alpha]^+ \wedge \Diamond[\beta]^+$

Let us call BSML^{lex} an analysis which assumes an obligatory application of $[\ ]^+$-enrichments in the scope of a modal. BSML^{lex} predicts a contrast between positive FC (valid) vs negative FC and modal disjunction (both not valid):

- Positive FC: $\Diamond[\alpha \vee \beta]^+ \models \Diamond[\alpha]^+ \wedge \Diamond[\beta]^+$
- Negative FC: $\neg\Box[\alpha \wedge \beta]^+ \not\models \neg\Box[\alpha]^+ \wedge \neg\Box[\beta]^+$
- Modal Disjunction: $\alpha \vee \beta \not\models \Diamond[\alpha]^+ \wedge \Diamond[\beta]^+$ (epistemic $\Diamond$)

As shown in Table 5, BSML^{lex}, modelling strong (i.e., obligatory) inferences, in combination with BSML^{*}, modelling global and weak (i.e., cancellable) effects, gives us a better match with the experimental findings than the assumption of an unrestricted global application of +-enrichment as implemented in BSML^{+}.

Similar lexically triggered local cases of pragmatic enrichment could be required for the interpretation of other expressions, for example marked indefinites, in particular those of the epistemic kind (Alonso-Ovalle and Menéndez-Benito, 2015). Consider the following examples of overt cancellations of the ignorance inference for plain disjunction, which in the current framework could be accounted for in terms of a neglect-zero suspension triggered by continuations like 'I know but I will not tell you' or 'Guess which':

(56) The prize is either in the attic or in the garden. I know that because I know where I put it, but I am not going to tell you. (from Grice 1989: 45).

(57) I was born in Tokyo or Kyoto. Guess which!

Epistemic indefinites are infelicitous in combination with such continuations. Consider German *irgend*-indefinites (Haspelmath, 1997; Kratzer and Shimoyama, 2002; Aloni and Port, 2015):

(58) Ignorance (epistemic indefinites, German)
a. *Irgendein* Student hat angerufen. #Rat mal wer?
Irgend-one student has called guess prt who?
b. Conventional meaning: Some student called – the speaker doesn't know who

We could account for this fact in a first order version of BSML assuming states to be sets of world-assignment pairs, $\langle w, g\rangle$, as in dynamic semantics. First, assume the following interpretation for the existential quantifier, where $s[x/d] = \{\langle w, g[x/d]\rangle \mid \langle w, g\rangle \in s\}$, and $g[x/d]$ is like g with the exception that $g[x/d](x) = d$.

(59) $M, s \models \exists x\phi$ iff there is $D^* \subseteq D$: $s = \cup_{d\in D^*} t_d$ & $M, t_d[x/d] \models \phi$

Then assume that *irgend*-indefinites, like modals in BSML^{lex}, necessarily trigger pragmatic enrichment in their scope and come with the additional anti-singleton requirement that the cardinality of D^* must be greater than 1 (Alonso-Ovalle and Menéndez-Benito, 2010).

(60) a. Irgendjemand called.
b. $\exists^{>1}x[\phi]^+$ [with $card(D^*) > 1$]

We would correctly predict the obligatory ignorance inferences triggered by these indefinites in episodic sentences (example (58)), but also their negative polarity behaviour under negation, obviation and distribution inferences under other operators and even their free choice readings (Kratzer and Shimoyama, 2002) for cases where $D^* = D$. Notice that in this framework differences between different indefinites could be accounted for in terms of different requirements with respect to (i) the cardinality of D^* and (ii) the obligatoriness of NE, which could characterise the difference between marked and unmarked indefinites.

Going back to the case of modal verbs, we observe that the following example is ok and it is a typical case of overt FC cancellation:

(61) You may have coffee or tea. Guess which!

How can this fact be reconciled with the predictions of BSML^{lex}? The difference between the epistemic indefinite (no overt cancellation possible) and the modal case is that for the latter, as explained in section 6.1, we can assume that the sluice forces a wide scope disjunction configuration, and in that case the obligatory enrichment triggered by the modal verb would not lead to a FC inference:

(62) a. You may have coffee or tea. Guess which!
b. $\Diamond[\alpha]^+ \vee \Diamond[\beta]^+ \not\models \Diamond\alpha$

Wide scope FC effects would still be captured as cancellable/global neglect-zero effects (with restrictions):

- BSML$^+$: $[\Diamond\alpha \vee \Diamond\beta]^+ \models_{BSML} \Diamond\alpha \wedge \Diamond\beta$ [if R is indisputable]
- BSML*: $\Diamond\alpha \vee \Diamond\beta \models_{BSML^*} \Diamond\alpha \wedge \Diamond\beta$ [if R is indisputable]

These are only tentative preliminary remarks but give rise to interesting testable predictions, for example on the strength of wide scope FC, which, even in the epistemic case where

			$\text{BSML}^{\emptyset}$	BSML^{lex}	BSML^{*}
NS FC	$\Diamond(\alpha \vee \beta) \rightsquigarrow \Diamond\alpha \wedge \Diamond\beta$	s	-	+	+
Dual Prohibition	$\neg\Diamond(\alpha \vee \beta) \rightsquigarrow \neg\Diamond\alpha \wedge \neg\Diamond\beta$	s	+	+	+
Negative FC	$\neg\Box(\alpha \wedge \beta) \rightsquigarrow \neg\Box\alpha \wedge \neg\Box\beta$	w	-	-	+
Modal disjunction	$\alpha \vee \beta \rightsquigarrow \Diamond\alpha \wedge \Diamond\beta$	w	-	-	+
WS FC	$\Diamond\alpha \vee \Diamond\beta \rightsquigarrow \Diamond\alpha \wedge \Diamond\beta$	?	-	-	+

Table 6: Comparison $\text{BSML}^{\emptyset}$, BSML^{lex} and BSML^{*}.

indisputability is assumed to hold, is predicted to be weaker than narrow scope FC.

The discussion so far has led us to a pluralism of systems definable in variants of BSML whose predictions are compared in Table 6:

- $\text{BSML}^{\emptyset}$: the NE-free fragment of BSML equivalent to classical modal logic, modelling logical-mathematical reasoning where neglect-zero effects are globally suspended;
- BSML^{lex}: modelling local neglect-zero effects, possibly due to the lexical meaning of some modal verbs; [⇒ obligatory?]
- BSML^{*}: modelling global, non-detachable neglect-zero effects. [⇒ cancellable]

I conjecture that these systems correspond to different interpretation strategies or reasoning styles people may adopt in different circumstances (e.g., ordinary conversation vs the context of a mathematical proof). As said, these are only preliminary remarks and these issues require future investigation, but I hope I succeeded in illustrating the potential of this framework to give rise to interesting empirically testable predictions, which as such can bring us a step further in understanding these long-standing issues.

8 Neglect-zero vs homogeneity

This final section briefly compares BSML with the homogeneity account of FC recently defended by Goldstein (2019). On Goldstein's proposal FC inferences are predicted for sentences of the form $\Diamond(\alpha \vee \beta)$ by assuming a homogeneity presupposition (roughly, either both $\Diamond\alpha$ and $\Diamond\beta$ are true or they are both false) added to the meaning of the possibility modal (in the alternative-based account, HAS) or of the disjunction (in the dynamic account, HDS). As observed by (Goldstein, 2019, pages 34-35) the predictions of BSML are close to those of his dynamic system, which next to Narrow Scope FC and Dual Prohibition, also accounts for Wide Scope FC at least for epistemic $\Diamond$.[28] Table 7 compares the validities of Goldstein's HAS and HDS with those of BSML^{+}, the pragmatically enriched version of BSML ($\alpha \models_{BSML^{+}} \beta$ iff $[\alpha]^{+} \models_{BSML} [\beta]^{+}$). As shown in the Table, HDS and BSML^{+} make the same predictions with respect to the core free choice facts, but differ in other respect: BSML^{+} validates transitivity but invalidates addition, LEM and contraposition, while for HDS the opposite holds. (Goldstein, 2019, pages 34-35) based his comparison on Aloni (2018), which did not define a general pragmatic enrichment function, but postulated as logical rendering of natural language 'or' an enriched disjunction $\vee_{+}$, defined as $(\phi \vee_{+} \psi) =: (\phi \wedge \text{NE}) \vee (\psi \wedge \text{NE})$. Aloni (2018) makes the same predictions as BSML^{+}, so in terms of empirical coverage (i.e. predictions with respect to free choice facts

[28]For the non-epistemic case, Wide Scope FC only holds in HDA if the accessibility relation R is assumed to satisfy the following condition that Goldstein calls universality: if a world v is accessible from any world then v is accessible from every world (see Goldstein, 2019, section 9.1). Universality is more restrictive than our state-based indisputability. For example, as far as I can see, universal frames would validate $\Diamond\alpha \models \Box\Diamond\alpha$ in HDA and this is not the case for indisputable frames in BSML.

		HAS	HDS	BSML$^+$
NS FC	$\Diamond(\alpha \vee \beta) \models \Diamond\alpha \wedge \Diamond\beta$	+	+	+
Dual Prohibition	$\neg\Diamond(\alpha \vee \beta) \models \neg\Diamond\alpha \wedge \neg\Diamond\beta$	+	+	+
WS FC (epistemic)	$\Diamond\alpha \vee \Diamond\beta \models \Diamond\alpha \wedge \Diamond\beta$	-	+	+
Modal Disjunction	$\alpha \vee \beta \models \Diamond\alpha \wedge \Diamond\beta$	-	+	+
Negative FC	$\neg\Box(\alpha \wedge \beta) \models \neg\Box\alpha \wedge \neg\Box\beta$	-	-	-
Addition	$\alpha \models \alpha \vee \beta$	+	+	-
LEM	$\models \alpha \vee \neg\alpha$	+	+	-
IAT	$\models (\Diamond\alpha \wedge \Diamond\beta) \vee (\neg\Diamond\alpha \wedge \neg\Diamond\beta)$	-	-	-
Upward Monotonicity	$\alpha \models \beta \;\Rightarrow\; \Diamond\alpha \models \Diamond\beta$	+	+	+
Transitivity	$\alpha \models \beta \;\&\; \beta \models \gamma \;\Rightarrow\; \alpha \models \gamma$	-	-	+
Contraposition	$\alpha \models \beta \;\Rightarrow\; \neg\beta \models \neg\alpha$	+	+	-

Table 7: Comparison with HAS and HDS from Goldstein (2019).

modulo wide scope FC for the non-epistemic case) these three accounts are all equivalent. One possible advantage of the present account over its two predecessors is that the FC potential / homogeneity-like status of disjunction here is not postulated but shown to follow from the cognitively plausible assumption that language users, when engaging in ordinary language interpretation, tend to neglect zero-models. If this assumption were confirmed, the present analysis would be more explanatory than (Goldstein, 2019) or (Aloni, 2018). A further potential advantage of a state-based approach over Goldstein's homogeneity account is that next to deriving detachable pragmatically enriched meanings (global BSML$^+$ and local BSML^{lex}), we can also represent classical literal meanings (BSML$^\emptyset$), and non-detachable neglect-zero effects (BSML*), which gives us an account of Negative FC. Furthermore, a conservative extension of BSML has been recently axiomatised by Anttila (2021), while the logical properties of Goldstein's adopted notion of Strawson entailment are unclear. One last remark on the logic side of things, Goldstein (2019) correctly observed that weak explosion fails in BSML ($p \wedge \neg p \not\models$ NE). However as mentioned earlier, BSML can also express a strong notion of contradiction (NE $\wedge \neg$NE) and explosion is sound for such a notion (NE $\wedge \neg$NE $\models \phi$).

9 Conclusion

Free choice inference represents a much discussed case of a divergence between logic and language. Grice influentially argued that the assumption that such divergence does in fact exist is a mistake originating "from inadequate attention to the nature and importance of the conditions governing conversation" (Grice 1989: 24). When applied to free choice phenomena, however, the standard implementation of Grice's view, modeling semantics and pragmatics as two separate components, has been shown to be empirically inadequate. We have proposed a different account: a bilateral state-based modal logic modelling next to literal meanings (the NE-free fragment), also pragmatic factors (NE) and the additional inferences that arise from their interaction (free choice and related inferences). The intruding pragmatic factor represented by NE, which connects to a version of Grice's Quality maxim, is a tendency of language users to neglect zero-models, i.e., models which verify sentences by virtue of the empty state, an abstract element comparable to the zero in mathematics. In terms of NE, we defined a pragmatic enrichment function and showed that, in interaction with disjunction occurring in positive contexts and *only* in these cases, pragmatic enrichment yields non-trivial effects including predicting narrow and wide scope FC inferences and their cancellation under negation:

- Narrow scope FC: $[\Diamond(\alpha \vee \beta)]^+ \models \Diamond\alpha \wedge \Diamond\beta$

BSML	Bilateral State Based Modal Logic	(Logical System)
BSML*	BSML without $\emptyset$	
BSML$^+$	BSML + global pragmatic enrichment	
BSML$^\emptyset$	BSML without NE	(= Classical Logic)
BSML^{lex}	BSML + local pragmatic enrichments (due to lexicalizations)	

Table 8: Overview of used labels.

- Wide scope FC: $[\Diamond\alpha \vee \Diamond\beta]^+ \models \Diamond\alpha \wedge \Diamond\beta$ [if R is indisputable]
- Dual Prohibition: $[\neg\Diamond(\alpha \vee \beta)]^+ \models \neg\Diamond\alpha \wedge \neg\Diamond\beta$
- Double Negation: $[\neg\neg\Diamond(\alpha \vee \beta)]^+ \models \Diamond\alpha \wedge \Diamond\beta$
- Modal Disjunction: $[\alpha \vee \beta]^+ \models \Diamond\alpha \wedge \Diamond\beta$ [if R is state-based]
- Detachability: $[\Diamond\neg(\neg\alpha \wedge \neg\beta)]^+ \not\models \Diamond\alpha \wedge \Diamond\beta$ [even if $\Diamond(\alpha \vee \beta) \equiv \Diamond\neg(\neg\alpha \wedge \neg\beta)$]

In the last part of the article we defined different variants of BSML (summarised in Table 8) and conjectured that they can be used to model different reasoning styles and interpretation strategies language users adopt in conversation. In particular we compared two ways to model neglect-zero effects in this framework:

- Syntactically, via the pragmatic enrichment function $[\ ]^+$ defined in terms of NE;
- Model-theoretically, by ruling out $\emptyset$ from the set of possible states $\mapsto$ BSML*

Both implementations derive FC effects (narrow and wide scope FC, the latter with restrictions) and their cancellations under negation (dual prohibition), but they also lead to empirical and conceptual differences: only BSML* defines a non-detachable notion of pragmatic inference and predicts Negative FC ($\neg\Box(\alpha \wedge \beta) \rightsquigarrow \neg\Box\alpha \wedge \neg\Box\beta$); and only in the original BSML, where the $\emptyset$ is one of the building blocks, locality (BSML^{lex}) and suspension (BSML$^\emptyset$) of neglect-zero effects can be modelled.

The contextual and cognitive factors playing a role in conversation are traditionally held to resist precise theorising. I hope with this research to have shown that logical methods can be fruitfully applied to model these factors and to make precise predictions about their impact on linguistic interpretation. In future work we plan to experimentally test these predictions; extend BSML with conditionals to study the effect of neglect-zero on the validity of a number of principles ruling their interpretation (including simplification of disjunctive antecedents); define multi-modal versions of BSML where the interplay between epistemic and deontic modality can be investigated in a rigorous way[29]; compare our relational notion of epistemic modality with other non-relational, state-based ones (e.g., $M, s \models \Diamond\phi$ iff $M, t \models \phi$, for some non-empty $t \subseteq s$) closer to the standard notion from dynamic semantics ($s[\Diamond\phi] = \{i \in s \mid s[\phi] \neq \emptyset\}$); and explore more consequences of our conjectured neglect-zero tendency and its cognitive plausibility.

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[29]This includes miners kind of puzzles (Kolodny and MacFarlane, 2010) but also, as suggested by an anonymous reviewer, "local" cases of wide scope FC in non-indisputable structures like in the inference from "maybe I can have soup or I can have salad, and maybe I can have neither' to "maybe I can have soup and I can have salad".

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