Logic and conversation: the case of free choice

Maria Aloni*

ILLC & Philosophy, University of Amsterdam

1 Introduction

The relation between literal meaning (semantics) and inferences based on language use (pragmatics) has been the subject of a longstanding debate in philosophy and linguistics and important progress has been made in the development of diagnostics to distinguish between semantic and pragmatic inference and in the formal derivation of the latter from general principles of conversation.[1] On the canonical view, pragmatic inference (aka conversational implicature) is derivable by general conversational principles, is cancellable, and is not embeddable under logical operators. Semantic inference, instead, lacks all these properties.

A number of inferences have been recently discussed in the literature which challenge the semantics vs pragmatics divide emerging from the canonical view. These include *ignorance inferences* in epistemic indefinites[2] and modified numerals[3] and phenomena of *free choice* (FC), where conjunctive meanings are unexpectedly derived for disjunctive sentences.[4] The common core of these inferences is that although they are derivable by conversational principles they lack at least one of the other defining properties of canonical pragmatic inference: they are often non-cancellable, they are sometimes embeddable, they are

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[1] Grice (1975, 1989); Gazdar (1979); Horn (1984); Levinson (1983, 2000); Sperber and Wilson (1995) and many others afterwards.

[2] E.g., Jayez and Tovena (2006); Alonso-Ovalle and Menéndez-Benito (2010, 2015); Aloni and Port (2010, 2015).

[3] E.g., Geurts and Nouwen (2007); Coppock and Brochhagen (2013); Schwarz (2016); Ciardelli *et al.* (2018a); van Ormondt (2019).

[4] Kamp (1973); Simons (2000); Zimmermann (2000); Geurts (2005); Klinedinst (2006); Aloni (2007); Fox (2007); Barker (2010) and many others.

acquired early and their processing time has been shown to equal that of literal interpretations (see Table 1 for illustrations). In this sense they are neither purely semantics nor purely pragmatics.

The overall goal of this project is to develop logics for these hybrid inferences that can capture their quasi-semantic behaviour while accounting for their pragmatic nature. The general strategy is to model such inferences as the result of the intrusion of pragmatic factors in the recursive process of meaning composition. In this sense these logics are meant to model, next to literal meanings, also pragmatic processes and the additional inferences derived by their interactions.

		pragm. derivable	**cancellable**	**non-embed.**	**proc. cost**	**acquisition**
Pragmatics	Conversational implicature J is always very punctual $\rightsquigarrow$ J is not a good philosopher	+	+	+	high	late
Semantics	Classical entailment I read some novels $\rightsquigarrow$ I read something	−	−	−	low	early
3rd Kind	Epistemic Indefinites *Irgendjemand* hat angerufen $\rightsquigarrow$ Speaker doesn't know who	+	−	+	?	?
	Modified Numerals Al has *at least two* degrees $\rightsquigarrow$ Maybe two, maybe more	+	−	+	?	?
	FC disjunction You may do A *or* B $\rightsquigarrow$ You may do *A*	+	?	?	low	early
	Scalar implicature I read some novels $\rightsquigarrow$ I didn't read all novels	+	+	?	high	late

Table 1: Beyond Gricean paradise

The present article focuses on the case of FC disjunction. In FC inference, conjunctive meanings are derived from disjunctive modal sentences contrary to the prescriptions of classical logic:

(1) $\Diamond(\alpha \vee \beta) \rightsquigarrow \Diamond\alpha \wedge \Diamond\beta$ (NB: $\Diamond\alpha \wedge \Diamond\beta \neq \Diamond(\alpha \wedge \beta)$)

Examples (2) and (3) illustrate cases of FC inference involving a deontic and an epistemic modal:

(2) Deontic FC (Kamp, 1973)

a. You may go to the beach or to the cinema.

b. $\rightsquigarrow$ You may go to the beach and you may go to the cinema.

(3) Epistemic FC (Zimmermann, 2000)

a. Mr. X might be in Victoria or in Brixton.
b. $\rightsquigarrow$ Mr. X might be in Victoria and he might be in Brixton.

Influential existing accounts view FC inferences as closely related to scalar implicatures (e.g., Fox, 2007; Chemla, 2008; Chierchia *et al.*, 2011; Franke, 2011; Bar-Lev and Fox, 2020). FC inferences and scalar implicatures, however, are different phenomena: as indicated in Table 1, they differ in their processing cost (Chemla and Bott, 2014) and in how early they are acquired (Tieu *et al.*, 2016). They further display different cancellability potential:

(4) I read some novels, in fact I read them all. (scalar)

(5) You may go to Paris or Rome, ??in fact you may not go to Paris. (FC)

And, finally, in contrast to scalar implicatures, FC inferences appear more to be part of *what is said* than of *what is merely implicated*, as shown by the following examples (modified from Mastop, 2005; Aloni, 2007):

(6) MOTHER: You may do your homework or help your father in the kitchen.
SON GOES TO THE KITCHEN.
FATHER: Go to your room and do your homework!
SON: But, mom said I could also help you in the kitchen.

(7) MOTHER: Al is French or Italian.
FATHER: She is both.
SON: ??But, mom said she is not both.

In view of these differences we conclude that FC inferences are not special cases of scalar implicatures: while scalar implicatures are naturally derived by applications of Grice's Quantity Maxim and typically rely on a comparison with a relevant set of alternatives, the operative pragmatic principle in our logic-based approach will be a version of Grice's Maxim of Quality and syntactic alternatives will play no role in our derivation. The next section introduces the phenomenon of FC and the challenges it involves for a logic-based account of linguistic meaning. Section 3 presents the core ingredients of the proposal; Section 4 defines the formal system; Section 5 presents the main results; Section 6 discusses a number of further applications and Section 7 concludes.[5]

2 The paradox of free choice

As mentioned in the introduction, sentences of the form "You may A or B" are normally understood as implying "You may A and you may B". The following, however, is not a valid principle in classical deontic logic (von Wright, 1968).

(8) $\Diamond(\alpha \vee \beta) \to \Diamond\alpha$ [Free Choice (FC) Principle]

As Kamp (1973) pointed out, plainly making the FC principle valid, for example by adding it as an axiom, would not do because it would allow us to derive any $\Diamond b$ from $\Diamond a$ as shown in (9):

(9) 1. $\Diamond a$ [assumption]

[5] Section 2 and parts of sections 4 and 6 are largely based on previous unpublished work by the author.

2. $\Diamond(a \vee b)$ [from 1, by classical reasoning]
3. $\Diamond b$ [from 2, by FC principle]

The step leading to 2 in the derivation above uses the following valid principle of classical modal logic:

(10) $\Diamond\alpha \rightarrow \Diamond(\alpha \vee \beta)$ [Modal Addition]

In natural language, however, (10) seems invalid: *You may post this letter* doesn't seem to imply *You may post this letter or burn it*, while (8) seems to hold, in direct opposition to the principles of deontic logic. Von Wright (1968) called this the paradox of free choice permission. Related paradoxes arise also for imperatives (see Ross' (1941) paradox), and other modal constructions.

Several solutions have been proposed to the paradox of free choice permission. Many have argued that what we called the FC principle is not a logical validity but a pragmatic inference derived as the product of rational interactions between cooperative language users rather than logic. The step leading to 3 in derivation (9) is therefore unjustified. Various ways of deriving free choice inferences as conversational implicatures have been proposed (e.g., Gazdar 1979, Kratzer and Shimoyama 2002, Schulz 2005, Fox, 2007 and Franke, 2011).

One argument in favour of a pragmatic account of FC comes from the observation that free choice effects disappear in negative contexts. For example, sentence (11) cannot merely mean that you cannot choose between the cake and the ice-cream as one would expect if free choice effects were semantic entailments rather than pragmatic implicatures (Alonso-Ovalle, 2006):

(11) Dual Prohibition
  a. You are not allowed to eat the cake or the ice cream.
  $\rightsquigarrow$ You are not allowed to eat either one.
  b. $\neg\Diamond(\alpha \vee \beta) \rightsquigarrow \neg\Diamond\alpha \wedge \neg\Diamond\beta$

Others have proposed modal logic systems where the step leading to 3 in (9) is justified but the step leading to 2 is no longer valid, e.g., Aloni (2007), who proposes a uniform semantic account of free choice effects of disjunctions and indefinites under both modals and imperatives.[6]

One argument in favor of a semantic approach comes from the observation that FC effects can embed under some operators, for example under universal quantifiers, as experimentally attested by Chemla (2009):

(12) Universal FC
  a. All of the boys may go to the beach or to the cinema.
  $\rightsquigarrow$ All of the boys may go to the beach and all of the boys may go to the cinema.
  b. $\forall x\Diamond(\alpha \vee \beta) \rightsquigarrow \forall x(\Diamond\alpha \wedge \Diamond\beta)$

Actually, examples like (12) are often used to argue against globalist accounts of implicatures, rather than in favor of semantic accounts. Notice however

[6] Simons (2005) and Barker (2010) also proposed semantic accounts of free choice inferences, the latter crucially employing an analysis of *or* in terms of linear logic additive disjunction combined with a representation of strong permission using the deontic reduction strategy as in Lokhorst (2006). While Aloni (2007) and Simons (2005) fail to account for Dual Prohibition cases, Barker (2010) derives it as a pragmatic inference.

that localists (Fox, 2007; Chierchia *et al.*, 2011) who predict the availability of embedded FC implicatures and therefore capture (12), need adjustments to capture the Dual Prohibition case illustrated in (11).

A third argument which can be used against most approaches mentioned so far comes from the observation that FC effects also arise with configurations where disjunction takes wide scope with respect to the modal (Zimmermann, 2000):

(13) Wide Scope FC
  a. Detectives may go by bus or they may go by boat. $\rightsquigarrow$ Detectives may go by bus and may go by boat.
  b. Mr. X might be in Victoria or he might be in Brixton. $\rightsquigarrow$ Mr. X might be in Victoria and might be in Brixton.
  c. $\Diamond\alpha \vee \Diamond\beta \rightsquigarrow \Diamond\alpha \wedge \Diamond\beta$

Wide scope FC inferences are hard to derive by Gricean means. Most pragmatic analyses of FC indeed only derive the narrow scope case (exceptions are Schulz (2005) and Chemla (2008)) and attempt to reduce all surface wide scope FC examples to cases of narrow scope FC:

(14) a. Detectives may go by bus or they may go by boat.
  $\rightsquigarrow$ Detectives may go by bus
  b. Logical Form: $\Diamond(\alpha \vee \beta)$ [rather than $\Diamond\alpha \vee \Diamond\beta$]

One argument against such a reductive strategy has been presented by Alonso-Ovalle (2006). Example (15) gives rise to a free choice inference, but an analysis of (15) as a narrow scope disjunction would require dubious syntactic operations (e.g., Simons' (2005) covert across-the-board (ATB) movement of the modal would not work here, because ATB movement requires identical modals in each clause):

(15) a. You may email us or you can reach the Business License office at 949 644-3141. $\rightsquigarrow$ You may email us
  b. Logical Form: $\Diamond\alpha \vee \Diamond\beta$ [it cannot be $\Diamond(\alpha \vee \beta)$]

One account which captures wide scope FC inference is the one defended by Zimmermann (2000) and further refined by Geurts (2005). Rather than attempting to develop an alternative to classical logic, Zimmermann (2000) criticises the standard logical translation of natural language *or* into Boolean disjunction $\vee$, and proposes instead a modal analysis of linguistic disjunction, which, as (16) illustrates, should be treated as a conjunctive list of epistemic possibilities rather than a Boolean disjunction ($\Diamond$ is here an epistemic possibility operator):

(16) A or $B \mapsto \Diamond\alpha \wedge \Diamond\beta$

Zimmermann then distinguishes between (13-c), which, according to him, is an unjustified logical principle, and the following intuitively valid natural language principle:

(17) X may A or may B $\rightsquigarrow$ X may A and X may B

By analysing disjunctions as conjunctions of epistemic possibilities, as in (16),

	NS FC	Dual Prohibition	Universal FC	WS FC
Semantic	yes	**no**	yes	?
Pragmatic				
global	yes	yes	**no**	**no**
local	yes	**no**	yes	**no**

Table 2: Semantic and pragmatic accounts of FC.

Zimmermann argues that the correct logical rendering of (17) is (18), which, if derived, explains our wide scope FC intuitions (again $\Diamond$ is an epistemic possibility operator and P is a deontic possibility operator):[7]

(18) $(\Diamond P\alpha \wedge \Diamond P\beta) \rightarrow (P\alpha \wedge P\beta)$

The account defended in this article shares the basic intuition of Zimmermann's and Geurts' analyses, namely that when one utters *A* or *B*, one normally conveys that each disjunct is an open epistemic possibility. My implementation of this idea, however, will be rather different from Zimmermann's or Geurts' and, as I hope to be able to show, it will be better equipped to capture the complex range of facts FC sentences give rise to.

To summarize, we have observed the following facts evidencing a hybrid behaviour of FC inferences in between semantics and pragmatics:

(19)
a. $\Diamond(\alpha \vee \beta) \rightsquigarrow \Diamond\alpha \wedge \Diamond\beta$ [Narrow Scope (NS) FC]
b. $\neg\Diamond(\alpha \vee \beta) \rightsquigarrow \neg\Diamond\alpha \wedge \neg\Diamond\beta$ [Dual Prohibition]
c. $\forall x\Diamond(\alpha \vee \beta) \rightsquigarrow \forall x(\Diamond\alpha \wedge \Diamond\beta)$ [Universal FC]
d. $\Diamond\alpha \vee \Diamond\beta \rightsquigarrow \Diamond\alpha \wedge \Diamond\beta$ [Wide Scope (WS) FC]

Several accounts have been proposed, which capture Narrow Scope FC or Wide Scope FC, but as summarized in Table 2, a purely semantic or pragmatic analysis typically fails to scale up to capture the whole range of relevant facts (at least without the help of additional *ad hoc* assumptions).[8]

In the following section I present a logic-based account of FC inference beyond the canonical semantics *vs* pragmatics divide. One of the main insights of Grice's "Logic and Conversation" was that pragmatically enriched meanings can be derived as the product of rational interactions between cooperative language users. The canonical implementation of this view assumes two separate components: semantics, on one side, ruled by classical logic; and pragmatics, on the other side, ruled by general principles of conversation. I propose a dif-

[7]Zimmermann actually only derives the weaker principle in (i) (under certain assumptions including his Authority principle):

(i) $(\Diamond P\alpha \wedge \Diamond P\beta) \rightarrow (\Box P\alpha \wedge \Box P\beta)$

$\Box\alpha$ should be read here as "it is certain that α". Geurts' (2005) refinement avoids this problem.

[8]A number of recent semantic accounts do much better than what is summarised in Table 2, for example Aher (2012) and Willer (2018) capture Dual Prohibition facts; Hawke and Steinert-Threlkeld (2018) account for Wide Scope FC (but only of the epistemic kind); Starr (2016) accounts for both Dual Prohibition and Wide Scope FC (but only of the deontic kind) and, finally, Goldstein (2019) for both Dual Prohibition and Wide Scope FC of the epistemic and deontic kind. Goldstein's dynamic homogeneity approach makes predictions which are very close to mine. I will comment on some of the differences at the end of Section 5.

ferent implementation of Grice's insight: I develop a non-classical (state-based) modal logic where (i) conversational principles can be modeled and can intrude in the recursive process of meaning composition, and (ii) an operation of pragmatic enrichment (denoted by a function $^+$) can be defined in terms of such intrusion. The resulting system derives FC and related inferences, but only for pragmatically enriched formulas, e.g.

$$\Diamond(\alpha \vee \beta)^+ \models \Diamond\alpha \wedge \Diamond\beta, \text{ but } \Diamond(\alpha \vee \beta) \not\models \Diamond\alpha \wedge \Diamond\beta$$

The upshot of a logic-based account is that the whole range of hybrid behaviour summarised in (19) can be naturally derived for pragmatically enriched sentences, while literal meanings ($^+$-free formulas) maintain their classical behaviour.

The next section introduces the main ingredients of the logical system I will employ to model pragmatic intrusion. It is a bilateral version of a state-based modal logic.

3 A logic for pragmatic intrusion

A state-based modal logic interprets formulas with respect to sets of possible worlds (states) rather than individual worlds. Let $M = \langle W, R, V \rangle$ be a classical Kripke model where W is a non-empty set of possible worlds, R an accessibility relation over W and V a world-dependent valuation function. Classical modal logic models *truth* in a possible world (an element of W). State-based modal logic models *support* in an information state (a subset of W):

- Classical modal logic:

$$M, w \models \phi, \text{ where } w \in W$$

- State-based modal logic:

$$M, s \models \phi, \text{ where } s \subseteq W$$

I will employ a *bilateral* version of a state-based modal logic which defines both support ($\models$) and anti-support ($\dashv$) conditions meant to capture the assertability and rejectability of a sentence in an information state.

- Bilateral state-based modal logic:

$$M, s \models \phi, \text{ “}\phi \text{ is assertable in } s\text{”}, \quad \text{with } s \subseteq W$$
$$M, s \dashv \phi, \text{ “}\phi \text{ is rejectable in } s\text{”}, \quad \text{with } s \subseteq W$$

In a state-based system, while logical consequence can be classical, we can still have states where neither ϕ nor $\neg\phi$ are supported.[9] See Figure 1 for an illustration.[10] Information states are then less determinate entities than possible worlds comparable to truthmakers (van Fraassen, 1969; Fine, 2017), possibilities (Humberstone, 1981; Holliday, 2015), or situations (Barwise and Perry, 1983).

[9]As in supervaluationism (van Fraassen, 1966), in a state-based system we can reject bivalence while validating the Law of Excluded Middle.

[10]In Figure 1, w_a stands for a world where only a is true, w_b only b, etc. These four possible worlds will be used for illustration throughout the paper.

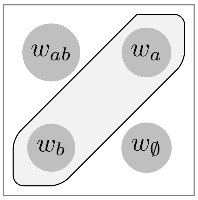


Figure 1: $M, s \not\models a$; $M, s \not\models \neg a$

The partial nature of an information state makes state-based systems particular suitable for capturing phenomena at the semantics-pragmatics interface, including anaphora (Groenendijk and Stokhof, 1991a; Groenendijk *et al.*, 1996; Dekker, 2012), questions (Ciardelli and Roelofsen, 2011; Ciardelli *et al.*, 2018b), epistemic modals (Veltman, 1996). The present article focuses on phenomena of FC. Crucial for this application is the adoption of a special notion of disjunction, called split or tensor disjunction, which has been studied in dependence/team logic (Yang and Väänänen, 2017), but see also (Cresswell, 2004; Hawke and Steinert-Threlkeld, 2018).[11]

Split Disjunction An information state s supports a split disjunction $\phi \vee \psi$ iff s is the union of (can be split into) two substates, each supporting one of the disjuncts:[12]

$$M, s \models \phi \vee \psi \quad \text{iff} \quad \exists t, t' : t \cup t' = s \ \& \ M, t \models \phi \ \& \ M, t' \models \psi$$

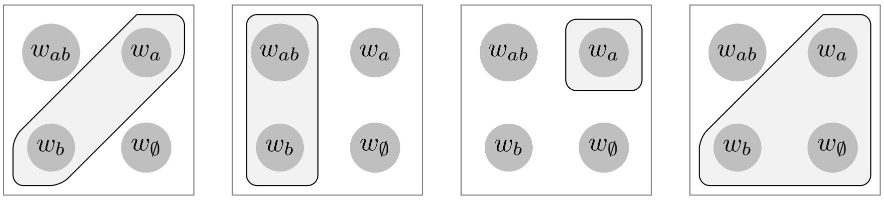
As an illustration consider the states represented in Figure 2. In these pictures again w_a stands for a world where only a is true, w_b only b, etc. The split disjunction $a \vee b$ is supported by the first three states, but not by 2(d). The latter contains $w_\emptyset$, a world where both a and b are false. No state containing such a world can be the union of two substates, each one supporting one of a and b. The state represented in 2(c), instead, does support $a \vee b$. This is because (i) according to the definition above, the substates necessary for verifying the disjunction can be empty, and (ii) in our logic the empty state supports all classical propositions. Therefore whenever a state s supports one classical disjunct, in this case a, we can find suitable substates of s supporting each classical disjunct: the state itself and the empty state.

[11] As far as I know, Hawke and Steinert-Threlkeld (2018) were the first to use split disjunction to explain FC phenomena. Their account however only applies to the case of epistemic modals, and, at least in this version, only captured the case of wide scope FC.

[12] There are at least two other ways to define disjunction in a state-based semantics (see Aloni, 2018, for a detailed comparison):

$$M, s \models \phi \vee_A \psi \quad \text{iff} \quad \forall w \in s : M, \{w\} \models \phi \text{ or } M, \{w\} \models \psi$$
$$M, s \models \phi \vee_B \psi \quad \text{iff} \quad M, s \models \phi \text{ or } M, s \models \psi$$

The first notion, $\vee_A$, corresponds to disjunction in possibility semantics and dynamic semantics (Humberstone, 1981; Groenendijk and Stokhof, 1991b). The second notion, $\vee_B$, is from inquisitive semantics (Ciardelli and Roelofsen, 2011) and some versions of truthmaker semantics (van Fraassen, 1969; Fine, 2017). All these notions collapse if s is a singleton, which corresponds to the classical case. Combined with an alternative-sensitive notions of modality (Aloni, 2007), $\vee_B$ derives *narrow* scope FC inference (Aloni and Ciardelli, 2013), but no wide scope FC.



(a) $\models (a \vee b)$; and $\models (a \vee b)^+$ (b) $\models (a \vee b)$; and $\models (a \vee b)^+$ (c) $\models (a \vee b)$; but $\not\models (a \vee b)^+$ (d) $\not\models (a \vee b)$; and $\not\models (a \vee b)^+$

Figure 2: Comparison $a \vee b$ and $(a \vee b)^+$

On my proposal, pragmatically enriched disjunctions crucially rule out the possibility of the empty state acting as one of the relevant substates for their evaluation: as we will see, a state s supports a pragmatically enriched disjunction $(\alpha \vee \beta)^+$ iff s is the union of two *non-empty* substates, each supporting one of the disjuncts. Pragmatically enriched disjunctions then require both disjuncts to be live possibilities, as in Zimmermann (2000) and Geurts (2005). Formula $(a \vee b)^+$ is no longer supported by the state represented in 2(c) because none of the worlds in the state verifies b (b is not a live possibility), so no non-empty substate of the state can be found which supports the second disjunct. The pragmatic enrichment function $^+$ is defined in terms of the so-called Non-Emptiness atom NE also from team logic (Yang and Väänänen, 2017).

Non-Emptiness atom (NE) Conversation is ruled by a principle that prescribes to avoid contradictions ('avoid $\bot$'). 'Avoid $\bot$' can be viewed as following from Grice's maxim of QUALITY: 'make your contribution one that is true'. The constant NE (Non-Emptiness) is on our proposal the formal counterpart of 'avoid $\bot$'.[13]

In a state-based semantics we have a possible state, the empty set $\emptyset$, which, as we said, vacuously supports every classical formula, including contradictions: $M, \emptyset \models p \wedge \neg p$. We may call $\emptyset$ the state of logical insanity. A consequence of this is that, while in classical logic there is no non-trivial way to model 'avoid $\bot$' (after all, $\neg\bot = \top$), in a state-based system we can model 'avoid $\bot$' as prescribing to avoid or neglect the state of logical insanity, or, equivalently, as requiring the supporting state to be *non-empty*. This is precisely the interpretation of the constant NE:[14]

$$M, s \models \text{NE} \quad \text{iff} \quad s \neq \emptyset$$

Given the characterisation of logical insanity in terms of the empty state, our analysis highlights a connection between the systematic compliance of language users with 'avoid $\bot$' and a tendency in human cognition to neglect abstract elements like the empty set (or the zero in mathematics, see Kaplan, 1999).

[13]The intuition behind NE is also connected to the idea behind the "inquisitive sincerity" and "attentive sincerity" maxims that have been discussed in (Roelofsen, 2013; Westera, 2013; Ciardelli *et al.*, 2014).

[14]As we will see in section 5, in BSML we can define different sorts of contradictions: classical contradiction like $p \wedge \neg p$, also referred to as weak contradiction, and a strong contradiction, $\text{NE} \wedge \neg\text{NE}$. Although $p \wedge \neg p$ and $\text{NE} \wedge \neg\text{NE}$ are not logically equivalent, NE expresses the negation of both in the following sense: $\neg\text{NE} \equiv (p \wedge \neg p)$ and $\text{NE} \equiv \neg(\text{NE} \wedge \neg\text{NE})$. Note, however, that $\text{NE} \not\equiv \neg(p \wedge \neg p)$. More on this in section 5.

Pragmatic enrichment ($^+$) The main insight of the present proposal is that FC and related inferences follow from a systematic "intrusion" of 'avoid $\bot$' (modeled as NE) in the recursive process of meaning composition. Such intrusion is formally implemented in terms of a pragmatic enrichment function $^+$ defined for the NE-free fragment of the language as follows (henceforth I will use α, β as metavariables ranging over formulas in the NE-free fragment of the language):

$$
\begin{array}{rcl}
p^+ & = & p \wedge \text{NE} \\
(\neg\alpha)^+ & = & \neg\alpha^+ \wedge \text{NE} \\
(\alpha \vee \beta)^+ & = & (\alpha^+ \vee \beta^+) \wedge \text{NE} \\
(\alpha \wedge \beta)^+ & = & (\alpha^+ \wedge \beta^+) \wedge \text{NE} \\
(\Diamond\alpha)^+ & = & \Diamond\alpha^+ \wedge \text{NE}
\end{array}
$$

As we will see, by enriching every formula with the requirement to satisfy NE (comply with 'avoid $\bot$') distributed along each of its subformulas, we derive narrow scope and wide scope FC inferences (the latter with restrictions) while no undesirable side effects obtain in other configurations, in particular under negation:

- Narrow scope FC: $\Diamond(\alpha \vee \beta)^+ \models \Diamond\alpha \wedge \Diamond\beta$
- Wide scope FC: $(\Diamond\alpha \vee \Diamond\beta)^+ \models \Diamond\alpha \wedge \Diamond\beta$ (with restrictions)
- Dual Prohibition: $\neg\Diamond(\alpha \vee \beta)^+ \models \neg\Diamond\alpha \wedge \neg\Diamond\beta$

The FC results rely on the adoption a notion of **modality** similar to the one employed in possibility semantics (Humberstone, 1981; Holliday, 2015), and inquisitive modal logic (Ciardelli, 2016)[15]. A state s supports a possibility modal sentence $\Diamond\phi$ iff for all worlds in s there is a non-empty subset of the set of worlds accessible from w which supports ϕ.

$$M, s \models \Diamond\phi \quad \text{iff} \quad \forall w \in s : \exists t \subseteq R[w] : t \neq \emptyset \ \&\ M, t \models \phi$$

The Dual Prohibition result relies on the adopted bilateralism, where each connective comes with an assertion ($\models$) and a rejection ($=\!\!|$) condition and **negation** is defined in terms of the latter notion (see Aher, 2012; Willer, 2018, who also adopt a bilateral notion of negation to capture Dual Prohibition):

$$
\begin{array}{lcl}
M, s \models \neg\phi & \text{iff} & M, s =\!\!| \ \phi \\
M, s =\!\!| \ \neg\phi & \text{iff} & M, s \models \phi
\end{array}
$$

The analysis resulting from our state-based account is not only empirically correct (e.g., some of the predictions have been experimentally confirmed, Cremers *et al.*, 2017; Tieu *et al.*, 2019), but also cognitively plausible. It is a common assumption that the interpretation of a sentence leads to the creation

[15] But different from the notion of modality employed in modal dependence logic (Väänänen, 2008). See (Anttila, 2021) for a comparison.

of a structure representing reality, a picture of the world (Johnson-Laird, 1983). This picture is inherently partial and is well represented by our information states. Our account shows that pragmatic inference of the free choice kind follows from the assumption that in creating such pictures language users systematically neglect the empty state. The empty state is an abstract construct which, we conjecture, speakers can access in certain circumstances (for example when engaging with logical reasoning) but is otherwise kept out of consideration.

On this view the systematic compliance of speakers to the principle 'avoid $\bot$' is a direct consequence of a general preference in human cognition for the concrete (a non-empty state) instead of the abstract (the empty set). This natural 'dispreference' for the empty set when applied to the nominal domain explains the existential import effects operative in the logic of Aristotle. When applied to the domain of possible worlds it explains the phenomena of free choice,[16] or at least this is my proposal.

The next section formally introduces Bilateral State-based Modal Logic (BSML).

4 Bilateral State-based Modal Logic (BSML)

Our target language L is the language of propositional modal logic enriched with the non-emptiness atom NE which as we said will be used to define the pragmatic enrichment function $^+$. Let A be a set of sentential atoms $A = \{p, q, \dots\}$.

Definition 1 (Language)

$$\phi \quad := \quad p \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \Diamond\phi \mid \text{NE}$$

where $p \in A$.

A Kripke model for L is a triple $M = \langle W, R, V \rangle$, where W is a set of worlds, R is an accessibility relation on W and V is a world-dependent valuation function for A.

Formulas in our language are interpreted in models M with respect to a state $s \subseteq W$. Both support, $\models$, and anti-support , $=\!|$, conditions are specified. $R[w]$ refers to the set $\{v \in W \mid wRv\}$.

[16] And the Kantian 'ought implies can' principle, as noted by Anttila (2021).

Definition 2 (Semantic clauses)

$$
\begin{array}{rll}
M, s \models p & \textit{iff} & \forall w \in s : V(w,p) = 1 \\
M, s =\!\mid p & \textit{iff} & \forall w \in s : V(w,p) = 0 \\[4pt]
M, s \models \neg\phi & \textit{iff} & M, s =\!\mid \phi \\
M, s =\!\mid \neg\phi & \textit{iff} & M, s \models \phi \\[4pt]
M, s \models \phi \vee \psi & \textit{iff} & \exists t, t' : t \cup t' = s \ \&\ M, t \models \phi \ \&\ M, t' \models \psi \\
M, s =\!\mid \phi \vee \psi & \textit{iff} & M, s =\!\mid \phi \ \&\ M, s =\!\mid \psi \\[4pt]
M, s \models \phi \wedge \psi & \textit{iff} & M, s \models \phi \ \&\ M, s \models \psi \\
M, s =\!\mid \phi \wedge \psi & \textit{iff} & \exists t, t' : t \cup t' = s \ \&\ M, t =\!\mid \phi \ \&\ M, t' =\!\mid \psi \\[4pt]
M, s \models \Diamond\phi & \textit{iff} & \forall w \in s : \exists t \subseteq R[w] : t \neq \emptyset \ \&\ M, t \models \phi \\
M, s =\!\mid \Diamond\phi & \textit{iff} & \forall w \in s : M, R[w] =\!\mid \phi \\[4pt]
M, s \models \textsc{ne} & \textit{iff} & s \neq \emptyset \\
M, s =\!\mid \textsc{ne} & \textit{iff} & s = \emptyset
\end{array}
$$

On the intended interpretation $M, s \models \phi$ stands for 'formula ϕ is assertable in s' and $M, s =\!\mid \phi$ stands for 'formula ϕ is rejectable in s', where s stands for the information state of the relevant speaker.

A state s supports an atomic proposition p iff p is true in all worlds in s; and anti-supports p iff p is false in all worlds in s. A state s supports a negation $\neg\phi$ iff it anti-supports ϕ; and anti-supports $\neg\phi$ iff it supports ϕ. A state s supports a disjunction $\phi \vee \psi$ iff s is the union of two substates, each supporting one of the disjuncts; and anti-supports $\phi \vee \psi$ iff s anti-supports ϕ and anti-supports ψ. A state s supports a conjunction $\phi \wedge \psi$ iff s supports ϕ and supports ψ; and anti-supports $\phi \vee \psi$ iff s is the union of two substates, each anti-supporting one of the disjuncts. A state s supports $\Diamond\phi$ iff for all $w \in s$: there is a non-empty subset of the set of worlds accessible from w which supports ϕ; and anti-supports $\Diamond\phi$ iff for all $w \in s$: the set of worlds accessible from w anti-supports ϕ. And finally a state s supports NE iff s is not empty; and it anti-supports NE iff it is empty.

We adopt the following abbreviation: $\Box\phi := \neg\Diamond\neg\phi$, and therefore easily derive the following interpretation for the necessity modal:

$$
\begin{array}{lll}
M, s \models \Box\phi & \text{iff} & \text{for all } w \in s : R(w) \models \phi \\
M, s =\!\mid \Box\phi & \text{iff} & \text{for all } w \in s : \text{there is a } t \subseteq R(w) : t \neq \emptyset \ \&\ t =\!\mid \phi
\end{array}
$$

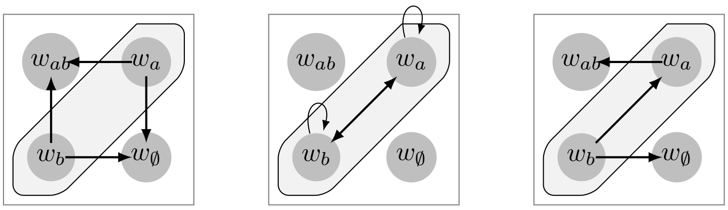
Logical consequence is defined as preservation of support.

Definition 3 (Logical consequence) $\phi \models \psi$ *iff for all* $M, s : M, s \models \phi \ \Rightarrow\ M, s \models \psi$

We also introduce a dependent notion of logical consequence, where we only consider model-state pairs (M, s) satisfying certain conditions.

Definition 4 (Logical consequence (restricted)) $\phi \models_X \psi$ *iff for all* $(M, s) \in X : M, s \models \phi \ \Rightarrow\ M, s \models \psi$

This restriction is used to express consequences which depend on "state-based" constraints on the accessibility relation R such as the ones defined in the following section.



(a) indisputable (but not state-based)

(b) state-based (and so also indisputable)

(c) neither

Figure 3: Indisputability vs state-basedness

4.1 State-based constraints on the accessibility relation

In a state-based modal logic we can formulate constraints on the accessibility relation in a model M which depend on a designated state in M. I will introduce two such constraints, indisputability and state-basedness, and later propose to employ these constraints to capture differences between epistemic and deontic modals.[17]

Definition 5 *Let* $M = (W, R, V)$ *and* $s \subseteq W$.

- *R is indisputable in (M, s) iff for all $w, v \in s : R[w] = R[v]$*
- *R is state-based in (M, s) iff for all $w \in s : R[w] = s$*

Indisputability relates to Zimmermann's (2000) Authority Principle. The property of being state-based relates to his Self-Reflection Principle.

An accessibility relation R is indisputable in a model-state pair (M, s) if any two worlds in s access exactly the same set of worlds according to R. Assuming s represents the information state of the relevant speaker, an indisputable R means that the speaker is fully informed about R, so, for example, if R represents a deontic accessibility relation, indisputability means that the speaker is fully informed about (or has full authority on) what is obligatory or allowed.

An accessibility relation R is state-based in a model-state pair (M, s) if all and only worlds in s are R-accessible within s. Trivially is R is state-based, R is also indisputable. The adoption of a state-based R will lead to the satisfaction of the classical S5 axioms but also to an account of the infelicity of so-called epistemic contradictions, i.e. the assertion of the epistemic possibility of a proposition ϕ conjoined with its negation (see section 6.1).

- Epistemic contradiction: $\Diamond\phi \wedge \neg\phi \models \bot$ [if R is state-based]

Epistemic contradiction only arises with epistemic modals. For this reason, we will assume a state-based R for epistemic modals but not for deontic ones:

1. Epistemic modal verbs: R is state-based

[17] The properties defined in Definition 5 are not closed under bisimulation and therefore they are not modally definable. For this reason Anttila (2021) proposes a different definition for these notions and discusses different possible characterisations. This issue however has no impact on the linguistic applications discussed in this paper. Therefore we present here the simpler (not modally definable) definitions.

2. Deontic modal verbs: R is possibly indisputable

An indisputable R instead yields wide scope FC inferences for pragmatically enriched formulas:

- Wide scope FC: $(\Diamond\alpha \vee \Diamond\beta)^+ \models \Diamond\alpha \wedge \Diamond\beta$ [if R is indisputable]

These facts together give us the following predictions: wide scope FC is always predicted for epistemic modals, which, leading to epistemic contradiction, require an accessibility relation which is state-based and therefore indisputable. Deontic modals instead only lead to wide scope FC inference in certain contexts, namely when the assumption of indisputability is justified. These are contexts where the speaker is assumed to be fully informed about what is obligatory or allowed, for example in performative uses of the verb. We will return to these predictions in section 6.2.

4.2 Pragmatic enrichment

The pragmatic enrichment function is recursively defined for formulas in the NE-free fragment of the language as follows:

Definition 6 (Pragmatic enrichment function)

$$\begin{array}{rcl} p^+ & = & p \wedge \text{NE} \\ (\neg\alpha)^+ & = & \neg\alpha^+ \wedge \text{NE} \\ (\alpha \vee \beta)^+ & = & (\alpha^+ \vee \beta^+) \wedge \text{NE} \\ (\alpha \wedge \beta)^+ & = & (\alpha^+ \wedge \beta^+) \wedge \text{NE} \\ (\Diamond\alpha)^+ & = & \Diamond\alpha^+ \wedge \text{NE} \end{array}$$

It is easy to see that pragmatic enrichment has a non-trivial effect on disjunctions. This effect in combination with our notion of modality allows us to derive FC inferences for pragmatically enriched formulas:

- Narrow scope FC: $(\Diamond(\alpha \vee \beta))^+ \models \Diamond\alpha \wedge \Diamond\beta$
- Wide scope FC: $(\Diamond\alpha \vee \Diamond\beta)^+ \models \Diamond\alpha \wedge \Diamond\beta$ [if R is indisputable]

A second crucial result is that pragmatic enrichment has non-trivial effects *only* on disjunctions, and only if they occur in a positive environment. In particular pragmatic enrichment is vacuous under negation, which allows us to derive Dual Prohibition:

- Dual Prohibition: $(\neg\Diamond(\alpha \vee \beta))^+ \models \neg\Diamond\alpha \wedge \neg\Diamond\beta$

Note however that in the scope of a double negation, FC effects arise again:

- Double Negation: $(\neg\neg\Diamond(\alpha \vee \beta))^+ \models \Diamond\alpha \wedge \Diamond\beta$

Evidence for the correctness of the latter prediction comes from cases of presupposition projection discussed by Romoli and Santorio (2019). Romoli and Santorio consider examples like (20) (where X_P means that X triggers the presupposition P):

(20) a. Either Maria can't go to study in Tokyo or Boston, or she is the first in our family who can go to study in Japan (and the second who can go to study in the States).

b. $\neg\Diamond(a \vee b) \vee X_{\Diamond a}$

In (20), the second disjunct *She is the first in our family who can go to study in Japan* presupposes *She can go to study in Japan*, but this presupposition does not project, it is filtered by the negation of the first disjunct. Assuming that a disjunction $A \vee B_P$ presupposes $\neg A \rightarrow P$, the predicted presupposition for (20) is (21):

(21) $\neg\neg\Diamond(a \vee b) \rightarrow \Diamond a$

In a system deriving FC inferences and validating Double Negation elimination (such as BSML but also Willer (2018); Goldstein (2019) and others) (21) is trivially satisfied (double negations cancel each other out and free choice inference is computed), so the correct filtering is predicted.

All these facts will be proven in the following section.

5 Results

5.1 Pragmatic enrichment, disjunction and modals

As mentioned in the previous sections, one of the main results of the present research is that pragmatic enrichment has a non-trivial effect in interaction with disjunction and only in interaction with disjunction. More precisely, we can prove the following two facts. First notice that from Definition 6 of Pragmatic Enrichment it directly follows that $\alpha^+ \models \text{NE}$, and an easy induction shows that $\alpha^+ \models \alpha$. So we have:

Fact 1 *Let α be NE-free.*

$$\alpha^+ \models \alpha \wedge \text{NE}$$

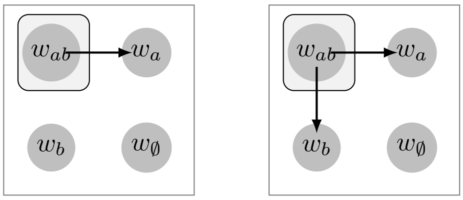
Another easy induction shows that if α does not contain any disjunction, then also the other direction holds:

Fact 2 *Let α be NE-free and $\vee$-free. Then*

$$\alpha \wedge \text{NE} \models \alpha^+$$

This means that if α is disjunction-free, the only difference between α and α^+ is that the former is supported by the empty set, $\emptyset$, while the latter is not. The effect of pragmatic enrichment in these cases is trivial in the sense that it does not lead to any linguistically interesting prediction. It is only in interaction with disjunction that pragmatic enrichment leads to non-trivial results and only when disjunction occurs in a positive context, as we will see in section 5.2. Let us have a closer look.

As we saw, in BSML a state s supports a plain disjunction iff s is the union of two substates, each supporting one of the disjuncts. The effect of pragmatically enriching $\alpha \vee \beta$ is that we add a conjunction with NE to each subformula of the original sentence. More precisely, $(\alpha \vee \beta)^+ = (\alpha^+ \vee \beta^+) \wedge \text{NE}$, which implies:



(a) $\models \Diamond(a \vee b)$; but $\not\models \Diamond(a \vee b)^+$

(b) $\models \Diamond(a \vee b)^+$

Figure 4: Narrow scope FC derived for pragmatically enriched disjunction

$$(\alpha \vee \beta)^+ \models (\alpha \wedge \text{NE}) \vee (\beta \wedge \text{NE})$$

Given the meaning of NE (only supported in non-empty states) the effect we obtain is that s supports $(\alpha \vee \beta)^+$ iff s is the union of two **non-empty** substates, each supporting one of the disjuncts. A pragmatically enriched disjunction $(\alpha \vee \beta)^+$ then requires both disjuncts to be live possibilities. See Figure 2 for illustrations. The notion of being a live possibility in a state can be expressed in BSML using the possibility modal $\Diamond$, if we assume a state-based accessibility relation.

Fact 3 (Modal disjunction) *Let S be the set of state-based model-state pairs.*

$$(\alpha \vee \beta)^+ \models_S \Diamond\alpha \wedge \Diamond\beta$$

Proof: Suppose $M, s \models (\alpha \vee \beta)^+$, which implies $M, s \models (\alpha \wedge \text{NE}) \vee (\beta \wedge \text{NE})$. The latter implies that there is a non-empty subset of s which supports α. By state-basedness, $R[w] = s$ for all $w \in s$, but then we have $M, s \models \Diamond\alpha$. By the same reasoning we conclude $M, s \models \Diamond\beta$, and therefore $M, s \models \Diamond\alpha \wedge \Diamond\beta$. □

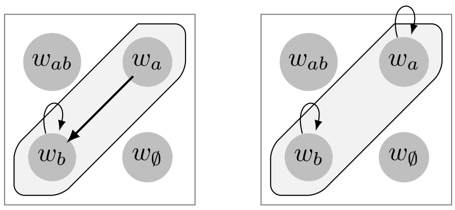
When embedded under a modal, a pragmatically enriched disjunction gives rise to FC effects.

Fact 4 (Narrow scope FC)

$$(\Diamond(\alpha \vee \beta))^+ \models \Diamond\alpha \wedge \Diamond\beta$$

Proof: Suppose $M, s \models (\Diamond(\alpha \vee \beta))^+$, which means $M, s \models \Diamond((\alpha^+ \vee \beta^+) \wedge \text{NE}) \wedge \text{NE}$, which implies $M, s \models \Diamond(\alpha^+ \vee \beta^+)$ and $s \neq \emptyset$. Let $w \in s$. $M, s \models \Diamond(\alpha^+ \vee \beta^+)$ means that there is a non-empty $t \subseteq R[w]$ such that $M, t \models (\alpha^+ \vee \beta^+)$. Therefore there are some t_1,t_2 such that $t = t_1 \cup t_2$ and $M, t_1 \models \alpha^+$ and $M, t_2 \models \beta^+$. Since, by Fact 1, $\alpha^+ \models \text{NE}$ and $\alpha^+ \models \alpha$, it follows that $t_1 \neq \emptyset$ and $M, t_1 \models \alpha$. Since w was arbitrary we conclude $M, s \models \Diamond\alpha$. By the same reasoning we conclude $M, s \models \Diamond\beta$ and therefore $M, s \models \Diamond\alpha \wedge \Diamond\beta$. □

Consider Figure 4 for an illustration. The pragmatically enriched $\Diamond(a \vee b)^+$ is not supported in the state depicted in 4(a) because $R[w_{ab}]$ does not support $(a \vee b)^+$ since b is not an open possibility there. The classical $(a \vee b)$, instead, is supported in $R[w_{ab}]$ with $\emptyset$ as relevant substate supporting b, and therefore the



(a) $\models \Diamond a \vee \Diamond b$, but $\not\models \Diamond a$

(b) $\models (\Diamond a \vee \Diamond b)^+$, but $\not\models \Diamond a$

Figure 5: Failures of wide scope FC.

state supports $\Diamond(a \vee b)$. In 4(b), instead, both a and b are open possibilities in $R[w_{ab}]$ and therefore $M, R[w_{ab}] \models (a \vee b)^+$ and so $M, s \models \Diamond(a \vee b)^+$.

From Fact 4 it directly follows that FC effects are also generated for disjunctions with logically dependent disjuncts and it is easy to see that the same holds for those with more than two disjuncts:

- $(\Diamond(\alpha \vee (\alpha \wedge \beta)))^+ \models \Diamond\alpha \wedge \Diamond(\alpha \wedge \beta)$
- $(\Diamond(\alpha \vee (\beta \vee \gamma)))^+ \models \Diamond\alpha \wedge \Diamond\beta \wedge \Diamond\gamma$

Wide scope FC instead is only derived if we assume an indisputable accessibility relation.

Fact 5 (Wide scope FC) *Let I be the set of indisputable model-state pairs.*

$$(\Diamond\alpha \vee \Diamond\beta)^+ \models_I \Diamond\alpha \wedge \Diamond\beta$$

Proof: Suppose $M, s \models (\Diamond\alpha \vee \Diamond\beta)^+$, which implies $M, s \models \Diamond\alpha^+ \vee \Diamond\beta^+$ and $s \neq \emptyset$. It follows that $s = t' \cup t''$ for some non-empty t' and t'' such that $M, t' \models \Diamond\alpha$ and $M, t'' \models \Diamond\beta$. Since R is indisputable, $R[w] = R[v]$ for all $w, v \in s$. But then $M, t' \models \Diamond\alpha$ implies $M, s \models \Diamond\alpha$ and $M, t'' \models \Diamond\beta$ implies $M, s \models \Diamond\beta$, and so $M, s \models \Diamond\alpha \wedge \Diamond\beta$. □

Pragmatic enrichment and indisputability are both needed to obtain wide scope FC inference, as illustrated in Figure 5. The accessibility relation in 5(a) is indisputable but wide scope FC fails since the disjunction is not pragmatically enriched and so the substate of s supporting $\Diamond a$ can be empty. In 5(b), instead, wide scope FC fails because R is not indisputable, and so while we have non-empty subsets of s supporting each disjunct, as required by the pragmatically enriched disjunction, s itself need not support them.

5.2 Negation, contradictions and pragmatic enrichments

In bilateral systems negation is taken to derive its meaning from the speech act of rejection, assumed to be a primitive notion (Smiley, 1996; Incurvati and Schlöder, 2017). In BSML rejection is represented by $\dashv$:

$$\begin{array}{lll} M, s \models \neg\phi & \text{iff} & M, s \dashv \phi \\ M, s \dashv \neg\phi & \text{iff} & M, s \models \phi \end{array}$$

One might argue that allowing to pre-encode what should happen under negation, bilateral systems like ours are more descriptive than explanatory (see Aloni, 2018; Bar-Lev and Fox, 2020). However our choices with respect to the anti-support clauses have not been arbitrary: if negation is grounded on the more primitive notion of rejection, it is speakers' behaviour which determines what happens under negation. Furthermore bilateral systems are not merely descriptive: their explanatory power comes with their overall predictions, which transcend the choices made with respect to the basic cases. In this subsection I will present some of these predictions, in particular those arising from the interaction between $\neg$ and our operation of pragmatic enrichment. But first I will discuss a number of general properties of negation in BSML.

First of all, it is easy to see that the current semantics validates a number of classical laws typical of Boolean negation ($\phi \equiv \psi$ is short for $\phi \models \psi$ and $\psi \models \phi$):

Fact 6 (Classical validities)

$$\begin{array}{rcll} \phi & \equiv & \neg\neg\phi & \textit{(Double Negation Elimination)} \\ \neg(\phi \vee \psi) & \equiv & \neg\phi \wedge \neg\psi & \textit{(De Morgan Laws)} \\ \neg(\phi \wedge \psi) & \equiv & \neg\phi \vee \neg\psi & \\ \neg\Box\phi & \equiv & \Diamond\neg\phi & \textit{(Duality } \Box/\Diamond\textit{)} \\ \neg\Diamond\phi & \equiv & \Box\neg\phi & \end{array}$$

Furthermore, our notion of negation is closely connected with the notion of incompatibility, a connection which has been identified as the hallmark of negation (Berto, 2015). In BSML we can show that if a state s supports a negative sentence $\neg\phi$, then s is incompatible with any state supporting ϕ, where being incompatible means having an empty intersection (for a proof see Anttila 2021, Proposition 3.3.9, page 53.)

Fact 7 (Negation and incompatibility)

$$M, s \models \neg\phi \;\Rightarrow s \cap t = \emptyset, \textit{ for all } t \textit{ such that } M, t \models \phi$$

Incompatibility however does not define negation in BSML. The other direction of the implication in Fact 7 does not hold. As a counterexample consider $\phi := \neg((p \wedge \text{NE}) \vee q)$. Let $\{w_q\}$ be a state consisting of a single world where q is true and p is false. Then $\{w_q\}$ is incompatible with any state supporting ϕ: $\{w_q\} \cap t = \emptyset$, for all t such that $M, t \models \phi$, but $M, \{w_q\} \not\models \neg\phi$, because $\neg\phi$ is equivalent to $(p \wedge \text{NE}) \vee q$, and p is not an open possibility in $\{w_q\}$.[18]

Our counterexample involves the non-emptiness atom NE.[19] In interaction with NE, bilateral negation gives rise to further non-classical behaviour, including a failure of replacement, and, therefore, a failure of the law of contraposition. As we will see, however, this non-classical behaviour is precisely what we need to explain the effects of pragmatic enrichment in negative contexts.

I will first show how bilateral negation leads to a failure of replacement in interaction with NE and then discuss how precisely this fact leads to correct predictions when bilateral negation applies to pragmatically enriched formulas.

[18] This means that to know the set of states which satisfy ϕ is not enough to know the set of states which satisfy $\neg\phi$. Therefore, bilateral negation is not a semantic operation in the sense of (Burgess, 2003; Kontinen and Väänänen, 2011).

[19] Other courterexamples, not involving NE, can be constructed in a language including the dependence atoms or inquisitive disjunction (Kontinen and Väänänen, 2011).

5.2.1 Tautologies and contradictions

In BSML we can distinguish between strong and weak notions of tautologies and contradictions:

- Weak
  - $\top : = \textsc{ne}$ — supported by all non-empty states
  - $\bot_1 : = p \wedge \neg p$ — supported only by the empty state
  - $\bot_2 : = \neg\textsc{ne}$ — supported only by the empty state
- Strong
  - $\boldsymbol{\top} : = p \vee \neg p$ — always supported
  - $\boldsymbol{\bot_1} := \textsc{ne} \wedge \bot_1$ — never supported
  - $\boldsymbol{\bot_2} := \textsc{ne} \wedge \bot_2$ — never supported

Weak contradictions $\bot_1$ and $\bot_2$ are mutually equivalent (supported only by the empty state) but behave differently under negation. Similarly for the strong contradictions $\boldsymbol{\bot_1}$ and $\boldsymbol{\bot_2}$.

- $\bot_1 \equiv \bot_2$, but $\neg\bot_1 \not\equiv \neg\bot_2$;
- $\boldsymbol{\bot_1} \equiv \boldsymbol{\bot_2}$, but $\neg\boldsymbol{\bot_1} \not\equiv \neg\boldsymbol{\bot_2}$.

These facts are examples of failures of replacement under negation.

Fact 8 (Failure of Replacement under $\neg$)

$$\phi \equiv \psi \not\Rightarrow \neg\phi \equiv \neg\psi$$

The different behaviour of weak and strong contradictions under negation is summarised in Figure 6. $\neg\bot_1$ is different from $\neg\bot_2$ because the former is equivalent to $\boldsymbol{\top}$, a strong tautology, while the latter is equivalent to $\top$, a weak tautology, and $\boldsymbol{\top} \not\equiv \top$ (the former is supported by any state, the latter only by all *non-empty* states). Similarly for $\boldsymbol{\bot_1}$ and $\boldsymbol{\bot_2}$:

- $\neg\bot_1 = \neg(p \wedge \neg p) \equiv p \vee \neg p = \boldsymbol{\top} \not\equiv \top = \textsc{ne} \equiv \neg\neg\textsc{ne} = \neg\bot_2$
- $\neg\boldsymbol{\bot_1} \equiv \neg(\textsc{ne} \wedge (p \wedge \neg p)) \equiv p \vee \neg p \equiv \boldsymbol{\top} \not\equiv \top = \textsc{ne} \equiv \neg(\textsc{ne} \wedge \neg\textsc{ne}) = \neg\boldsymbol{\bot_2}$

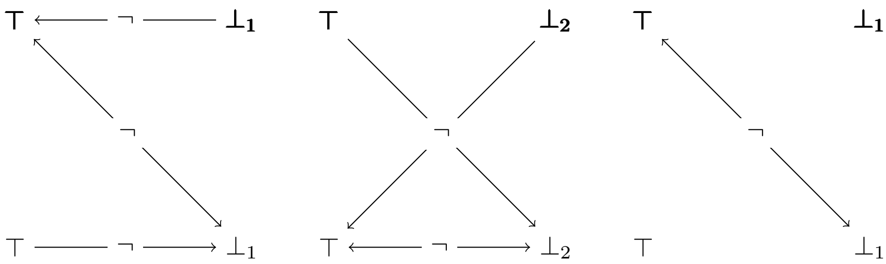
Furthermore, although strong and weak tautologies are different, their negations are equivalent:

- $\neg\boldsymbol{\top} = \neg(p \vee \neg p) \equiv p \wedge \neg p \equiv \neg\textsc{ne} = \neg\top$

In combination with the soundness of double negation elimination this leads (again) to a failure of replacement under $\neg$:

- $\neg\boldsymbol{\top} \equiv \neg\top$, but $\neg\neg\boldsymbol{\top} \not\equiv \neg\neg\top$

The cause of all problems seems to be NE when occurring under negation. For this reason in dependence/team logic negation is usually only defined for the classical fragment of the language. This would give us the "well-behaving" Negation Square 3 in Figure 6(c). But this strategy is not an option for us. The effect of pragmatic enrichment (formulated in terms of NE) in negative sentences is among the phenomena that motivated this research. In what follows I will show that despite (or better because of) the logical misbehaviour of NE under $\neg$, our predictions with respect to pragmatic enrichments of negative sentences are in agreement with our natural language intuitions.



(a) Negation Square 1 (b) Negation Square 2 (c) Negation Square 3

Figure 6: Negation Squares

5.2.2 Pragmatic enrichment under negation

First of all, we have already seen that FC effects disappear under negation (Dual Prohibition facts), so pragmatic enrichment should be vacuous in such an environment:

$$\neg\alpha^+ \equiv \neg\alpha$$

On the other hand, as Romoli and Santorio (2019) argued, there is evidence showing that speakers draw FC inferences under double negation (Double Negation facts), so pragmatic enrichment should not be vacuous there:

$$\neg\neg\alpha^+ \not\equiv \neg\neg\alpha$$

Without replacement failure these two desiderata would be impossible to satisfy.

Before proving these facts a clarification: the pragmatic enrichment of a negative sentence is defined as follows:

$$(\neg\alpha)^+ = \neg\alpha^+ \wedge \text{NE}$$

When we say that pragmatic enrichment is vacuous under negation we do not mean to say that pragmatically enriching a negative sentence is always trivial. For example, a pragmatically enriched negative disjunct can give rise to free choice effects:

$$(\neg\alpha)^+ \vee \psi \not\equiv (\neg\alpha \vee \psi)$$

What we mean instead is that if α is a positive sentence occurring under a negation, pragmatically enriching α is vacuous there.

Fact 9 (Pragmatic enrichment vacuous under single negation) *Let α be a positive sentence, then*

$$\neg\alpha^+ \equiv \neg\alpha$$

Proof: By induction on the complexity of α. We only give two instructive cases:

- $\alpha = p$. $\neg p^+ \equiv \neg(p \wedge \text{NE}) \equiv \neg p \vee \neg\text{NE} \equiv \neg p$
- $\alpha = \beta \vee \gamma$. $\neg(\beta \vee \gamma)^+ \equiv \neg((\beta^+ \vee \gamma^+) \wedge \text{NE}) \equiv \neg(\beta^+ \vee \gamma^+) \vee \neg\text{NE} \equiv \neg(\beta^+ \vee \gamma^+) \equiv \neg\beta^+ \wedge \neg\gamma^+ \equiv_{\text{IH}} \neg\beta \wedge \neg\gamma \equiv \neg(\beta \vee \gamma)$

The crucial fact exploited in the proof of Fact 9 is that $\neg(\phi \wedge \textsc{ne})$ is by the de Morgan laws equivalent to $\neg\phi \vee \neg\textsc{ne}$, which, since only the empty set supports $\neg\textsc{ne}$, is equivalent to $\neg\phi$. Note however that if $\neg(\phi \wedge \textsc{ne})$ occurs under another negation, things change. From $\neg\neg(\phi \wedge \textsc{ne})$, by double negation elimination, we obtain $(\phi \wedge \textsc{ne})$ which is not equivalent with ϕ or $\neg\neg\phi$. Pragmatic enrichment then is not vacuous under double negation.

Fact 10 (Pragmatic enrichment not vacuous under double negation)

$$\neg(\neg\alpha)^+ \equiv \neg\neg\alpha^+ \not\equiv \neg\neg\alpha$$

Proof: Here is a counterexample:

$$\neg(\neg p)^+ \equiv \neg(\neg p^+ \wedge \textsc{ne}) \equiv \neg\neg p^+ \vee \neg\textsc{ne} \equiv \neg\neg p^+ \equiv \neg\neg(p \wedge \textsc{ne}) \equiv p \wedge \textsc{ne} \not\equiv p \equiv \neg\neg p$$

So we have another example of a failure of replacement under negation: $\neg p^+ \equiv \neg p$ (Fact 9), but $\neg\neg p^+ \not\equiv \neg\neg p$ (Fact 10).

As a corollary of Fact 9 we obtain Dual Prohibition:

Fact 11 (Dual Prohibition)

$$(\neg\Diamond(\alpha \vee \beta))^+ \models \neg\Diamond\alpha \wedge \neg\Diamond\beta$$

Double Negation follows from Fact 4 and Double Negation Elimination:

Fact 12 (Double Negation)

$$(\neg\neg\Diamond(\alpha \vee \beta)^+ \models \Diamond\alpha \wedge \Diamond\beta$$

Another prediction of BSML is that logically equivalent sentences can have different pragmatic effects:

Fact 13 (Detachability)

$$\begin{array}{rcl} \neg\alpha \vee \neg\beta & \equiv & \neg(\alpha \wedge \beta) \\ (\neg\alpha \vee \neg\beta)^+ & \not\equiv & (\neg(\alpha \wedge \beta))^+ \end{array}$$

Although $\neg\alpha \vee \neg\beta$ and $\neg(\alpha \wedge \beta)$ are logically equivalent only disjunctions give rise to FC effects, which means that on our account FC inferences are detachable:

$$\begin{array}{rcl} (\Diamond(\neg\alpha \vee \neg\beta))^+ & \models & \Diamond\neg\alpha \\ (\Diamond\neg(\alpha \wedge \beta))^+ & \not\models & \Diamond\neg\alpha \\ (\neg\Box(\alpha \wedge \beta))^+ & \not\models & \Diamond\neg\alpha \end{array}$$

Our predictions here differ from those of the implicature account of free choice (e.g., Fox, 2007), but also from Willer (2018). Instead they match the predictions of other inquisitive accounts (Aher, 2012; Ciardelli *et al.*, 2018c), who have argued for their correctness based on examples like (22) from (Ciardelli *et al.*, 2018c):

(22) a. Mary might not speak both Arabic and Bengali.
b. #So, she might not speak Arabic.

Note however that in contrast to inquisitive accounts, BSML needs not assume a failure of the de Morgan Laws to account for these cases.

Let me conclude this section with a mention of the homogeneity account of FC recently defended by Goldstein (2019). On Goldstein's account FC inferences are predicted for sentences of the form $\Diamond(A \vee B)$ by assuming a homogeneity presupposition (roughly, either both $\Diamond A$ and $\Diamond B$ are true or they are both false) added to the meaning of the possibility modal (alternative-based account) or of the disjunction (dynamic account). As observed by Goldstein, the predictions of BSML are very close to those of his dynamic system, which next to Narrow scope FC and Dual Prohibition, also accounts for Wide Scope FC (with restrictions). Goldstein based his comparison on Aloni (2018), which did not define a general pragmatic enrichment function, but postulated as logical rendering of natural language 'or' an enriched disjunction $\vee_+$, defined as $(\phi \vee_+ \psi) =: (\phi \wedge \text{NE}) \vee (\psi \wedge \text{NE})$. Aloni (2018) makes the same predictions as the present system, so in terms of empirical coverage these accounts are all equivalent. I see however one important advantage of the present account: the FC potential / homogeneity-like status of disjunction here is not postulated but shown to follow from the systematic intrusion of 'avoid $\bot$' in the process of interpretation, an intrusion which, as we saw, is connected to the cognitively plausible assumption that language users, when engaging in ordinary conversation, tend to neglect the empty state. In this sense the present analysis is more explanatory than (Goldstein, 2019) or (Aloni, 2018). A further advantage of a state-based approach over Goldstein's homogeneity account is that next to representing pragmatically enriched meanings we can also recover classical literal meanings. The NE-free fragment of the language indeed behaves classically:

Fact 14 *Let $\models_C$ denote logical consequence as defined in classical modal logic. Then for* NE*-free α, β:*

$$\alpha \models \beta \Leftrightarrow \alpha \models_C \beta$$

Furthermore, conceptually, the origin and status of Goldstein's assumed homogeneity presupposition is unclear as well as the logical properties of his adopted notion of Strawson entailment. A conservative extension of BSML instead has been recently axiomatised by Anttila (2021). One last remark on the logic side of things, Goldstein (2019) correctly observes that weak explosion fails in BSML ($p \wedge \neg p \not\models \text{NE}$). However as mentioned earlier, BSML can also express a strong notion of contradiction ($\text{NE} \wedge \neg\text{NE}$) and explosion is sound for such a notion ($\text{NE} \wedge \neg\text{NE} \models \phi$).

6 Further Applications

6.1 Epistemic contradiction

Wittgenstein observed the illegitimacy of asserting both "it might be that ϕ" and "it is not the case that ϕ" in a single context (see also Veltman, 1996; Hawke and Steinert-Threlkeld, 2018; Mandelkern, 2017, and others):

(23) #It might be raining but it is not raining.

Yalcin (2007) called these sentences *epistemic contradictions.* The challenge for a logic-based account is to derive the infelicity or incoherence of (23) ($\Diamond\phi\wedge\neg\phi \models$

$\bot$), while preserving the non-factivity of the $\Diamond$ operator ($\Diamond\phi \not\models \phi$). In BSML, we predict exactly this behaviour if we assume that the relevant accessibility relation is state-based.

Fact 15 (Epistemic contradiction) *Let S be the set of model-state pairs (M, s) such that R is state-based in (M, s). Let $\bot \in \{\bot_1, \bot_2\}$.*

$$\neg\phi \wedge \Diamond\phi \models_S \bot$$

Proof: Suppose $M, s \models \neg\phi \wedge \Diamond\phi$. If R is state-based in (M, s), then $M, s \models \Diamond\phi$ implies that either (i) $s = \emptyset$ or (ii) there is a non-empty subset t of s such that $M, t \models \phi$. By Fact 7, $M, s \models \neg\phi$ implies that $s \cap t = t = \emptyset$. Option (ii) is then impossible, $s = \emptyset$, and therefore $M, s \models \bot$.

Fact 16 (Non factivity) *Let S be the set of model-state pairs (M, s) such that R is state-based in (M, s).*

$$\Diamond\phi \not\models_S \phi$$

Proof: A counterexample is given in Figure 3(b).

As already mentioned in section 4.1, in BSML the difference between different modalities can be captured in terms of differences in properties of the accessibility relation. Assuming a state-based R for epistemic modals but not for deontic modals, we correctly predict that only the former lead to epistemic contradictions:

(24) #It might be raining and it is not raining.

(25) You are not there but you may go there.

The assumption that epistemic modals trigger state-based R has consequences also for their free choice potential, as we will see in the following subsection.

6.2 Epistemic and Deontic Free Choice

As we saw, BSML derives both narrow scope and wide scope FC effects for pragmatically enriched sentences, but while narrow scope effects are generated for any modality, wide scope FC arises only in case the modality is of the indisputable kind.

1. $\Diamond(\alpha \vee \beta)^+ \models \Diamond\alpha \wedge \Diamond\beta$
2. $(\Diamond\alpha \vee \Diamond\beta)^+ \models \Diamond\alpha \wedge \Diamond\beta$ [if R is indisputable]

Since state-based R are also indisputable, narrow and wide scope FC are always predicted for epistemics, which involve state-based R:

(26) He might either be in London or in Paris. [+fc, narrow]

(27) He might be in London or he might be in Paris. [+fc, wide]

The case of deontic FC is more subtle. Assuming that deontics trigger an indisputable R only in certain contexts, namely when the speaker is assumed to be knowledgeable about what is permitted/obligatory (e.g. in performative uses), we make the following predictions:

- narrow scope FC is always predicted for deontics
- wide scope FC is predicted only if speaker knows what is permitted/obligatory

These predictions have received preliminary confirmations from experiments reported in Cremers *et al.* (2017). In these experiments, judgements on free choice effects were collected, for both wide and narrow scope disjunctions, in different contexts with the speaker assumed to be knowledgeable or not. To distinguish narrow scope from wide scope configurations, examples like the following were used, where the position of *either* arguably constrains the syntactic scope of *or*:

(28) We may either eat the cake or the ice cream. [narrow scope disjunction favoured, but not forced]

(29) Either we may eat the cake or the ice cream. [wide scope disjunction forced]

More specifically, as argued by Larson (1985), the high position of *either* in (29) forces a wide scope disjunction configuration, while its low position in (28) favours a narrow scope interpretation. One rather surprising result of these experiments was that only in wide scope configurations like (29) the availability of the FC inference was dependent on the assumption on speaker knowledge, exactly as predicted by the present analysis.

A further consequence of our analysis is that all cases of overt free choice cancellation must be treated as examples of wide scope disjunction. This is arguably the case for examples like (30) involving sluicing, as discussed in Fusco (2019):

(30) You may eat the cake or the ice-cream, I don't know which.

Whether the same assumption is also justified in cases like (31), discussed by Kaufmann (2016), must be left to another occasion:

(31) You may either eat the cake or the ice-cream, it depends on what John has taken.

If we assumed that sluicing always requires wide scope disjunctions as antecedents, the first sentence in (32) would also be a case of a wide scope disjunction:

(32) You may either eat the cake or the ice-cream, I don't care which.

But then we would predict that FC inferences would be generated only in contexts where the speaker is assumed to be knowledgeable and this prediction does not seem to be correct. (32) seems to trigger a FC inference no matter what. Notice however that as Fusco argues there is a difference in the elided material of (30) and (32), as illustrated by the following pair:

(33) You may either eat the cake or the ice-cream, I don't know which you may eat.

(34) You may either eat the cake or the ice-cream, I don't care which you eat.

But then we can assume that the sluicing construction in (32) requires a disjunction of the form 'You eat the cake or you eat the ice-cream' as antecedent (rather than 'You may eat the cake or you may eat the ice-cream') and so triggers a narrow scope disjunction configuration in the first sentence, which in our system always gives rise to free choice effects.

6.3 Ignorance and Obviation

While all human languages appear to contain a word for negation, there are various examples of languages lacking explicit coordination structures. In these languages there is no word corresponding to *or*, but disjunctive meanings can typically still be expressed for example by adding a suffix/particle expressing uncertainty to the main verb. Example (35) illustrates this strategy for Maricopa (a Yuman language of Arizona described by Gil (1991)):

(35) Johnš Billš v?aawuumšaa.
John-nom Bill-nom 3-come-pl-fut-infer
'John or Bill will come'

(36) Johnš Billš v?aawuum.
John-nom Bill-nom 3-come-pl-fut
'John and Bill will come' [Maricopa, Gil 1991, p. 102]

In (35) the "uncertainty" suffix *šaa* is added to the main verb and it is what triggers a disjunctive interpretation. Indeed when omitted as in (36) the interpretation of the sentence becomes conjunctive. This example provides evidence of the close connection between disjunction and uncertainty. Plain disjunctions give rise to ignorance effects (Gazdar, 1979):

(37) Ignorance
a. John has two or three children.
$\rightsquigarrow$ speaker doesn't know how many
b. $\alpha \vee \beta \rightsquigarrow \Diamond\alpha \wedge \Diamond\beta$ [epistemic $\Diamond$]

These effect are quite strong as evidenced by the oddity of sentences like (38):

(38) ?I have two or three children.

(38) is odd because it suggests that the speaker doesn't know how many children she has, an implausible assumption. The ignorance effect (and therefore the oddity) disappear when we embed the disjunction under a universal quantifier:

(39) Obviation
a. Every woman in my family has two or three children.
$\not\rightsquigarrow$ speaker doesn't know how many children each woman has
b. $\forall x(\alpha \vee \beta) \not\rightsquigarrow \forall x(\Diamond\alpha \wedge \Diamond\beta)$ [epistemic $\Diamond$]

Disjunctions under universal quantifiers have been argued instead to give rise to a distribution inference (Fox, 2007; Klinedinst, 2006):

(40) Distribution

a. Every woman in my family has two or three children.
$\rightsquigarrow$ some woman has two and some woman has three

b. $\forall x(\alpha \vee \beta) \rightsquigarrow \exists x\alpha \wedge \exists x\beta$

The challenge here is to account for the ignorance inference in (38) and its obviation in (39), as well as the distribution effect in (40). As we saw, BSML derives modal effects for pragmatically enriched plain disjunctions, which in combination with an exclusivity implicature, derives the desired ignorance inference:

- $(\alpha \vee \beta)^+ \models \Diamond\alpha \wedge \Diamond\beta$ [if R is state-based]

van Ormondt (2019) discusses a first order extension of BSML, where states are defined as sets of world-assignment pairs, which also accounts for the obviation and distribution cases in (39) and (40).

7 Conclusion

Free choice inference represents a much discussed case of a divergence between logic and language. Grice influentially argued that the assumption that such divergence does in fact exist is a mistake originating "from inadequate attention to the nature and importance of the conditions governing conversation" (Grice 1989: 24). When applied to free choice phenomena, however, the standard implementation of Grice's view, modeling semantics and pragmatics as two separate components, has been shown to be empirically inadequate. I proposed a different implementation of Grice's insight: a bilateral state-based modal logic modelling next to literal meanings (the NE-free fragment, ruled by classical logic), also pragmatic principles (NE) and the additional inferences that arise from their interaction (free choice and related inferences). The intruding pragmatic principle represented by NE, a version of Grice's Quality maxim, has been shown to connect to a tendency of language users to neglect the empty state, an abstract element comparable to the zero in mathematics. In terms of NE, we defined a pragmatic enrichment function and showed that, in interaction with disjunction occurring in positive contexts and *only* in these cases, pragmatic enrichment yields non-trivial effects including predicting narrow and wide scope FC inferences and their cancellation under negation:

- Narrow scope FC: $(\Diamond(\alpha \vee \beta))^+ \models \Diamond\alpha \wedge \Diamond\beta$
- Wide scope FC: $(\Diamond\alpha \vee \Diamond\beta)^+ \models \Diamond\alpha \wedge \Diamond\beta$ [if R is indisputable]
- Dual Prohibition: $(\neg\Diamond(\alpha \vee \beta))^+ \models \neg\Diamond\alpha \wedge \neg\Diamond\beta$
- Double Negation: $(\neg\neg\Diamond(\alpha \vee \beta))^+ \models \Diamond\alpha \wedge \Diamond\beta$
- Modal Disjunction: $(\alpha \vee \beta)^+ \models \Diamond\alpha \wedge \Diamond\beta$ [if R is state-based]
- Detachability: $(\Diamond\neg(\neg\alpha \wedge \neg\beta))^+ \not\models \Diamond\alpha \wedge \Diamond\beta$
[even if $\Diamond(\alpha \vee \beta) \equiv \Diamond\neg(\neg\alpha \wedge \neg\beta)$]

In future work we plan to experimentally test these predictions; further develop the first order version of BSML and its possible applications (e.g., to modified numerals and marked indefinites); and explore more consequences of our conjectured 'neglect $\emptyset$' tendency and its cognitive plausibility.

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